Loan guarantees, bank underwriting policies and financial fragility*

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Abstract

Loan guarantees often represent a form of government intervention designed to support bank lending and avoid credit market freezes. However, their use has given rise to concerns as to how they may affect banks' risk-taking incentives. In a model of financial fragility that incorporates bank capital and a portfolio choice problem, we analyze the effects of loan guarantees on banks' underwriting standards and run risk. We show that loan guarantees reduce run risk and improve banks' underwriting standards except possibly for poorly capitalized banks. We highlight a novel feedback effect between banks' underwriting choices and depositors' run decisions.

Keywords: panic runs, fundamental runs, bank monitoring, charter value

JEL classifications: G21, G28

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1 Introduction

Periods of crisis, where economic fundamentals are poor as a result of exogenous shocks – either temporarily or more long-term – are catalysts for government intervention. Often these crises are coupled with credit market freezes, with banks sitting on capital rather than lending it out, possibly further worsening fundamentals to the extent that viable firms get denied credit. This creates a need for government support of bank lending activities in order to minimize disruptions to the real economy. A case in point is the Covid-19 pandemic, which erupted in early 2020 as an unexpected and exogenous shock leading to a sudden and deep liquidity crisis for non-financial corporates and triggering massive interventions by public authorities.

Despite differences across countries, one major form of intervention were public guarantee schemes (PGSs) aimed at sustaining bank lending by providing a guarantee on bank loans.[1] To give an order of magnitude, in Europe more than 320 billion euros of new loans were provided under PGSs in the four major European countries (France, Germany, Italy and Spain) as of September 2020 (ECB, 2020). Similarly, in the US 5.16 million borrowers had access to guaranteed loans through the 669 billion dollars Paycheck Protection Program (PPP) as of end of November 2020 (Balyuk, Prabhala, and Puri, 2020).

Notwithstanding their possible effectiveness as a stimulative tool, the use of loan guarantees raises a number of important questions in terms of their implications for banks' underwriting processes and thus, ultimately, for financial stability.[2] As with any form of insurance, the introduction of PGSs may reduce banks' incentives to select and monitor borrowers properly. Indeed, this is a commonly held view concerning the impact of deposit insurance, for instance, and is often cited as a rationale for macro-prudential policies (e.g., capital requirements) to control excessive risk-taking. This perception has fueled a debate concerning the impact of PGSs and whether the reliance on public support programs induces banks to engage in "evergreening," thus keeping nonviable firms alive (see, e.g., Acharya, Borchert, Jager, and Steffen, 2020; Acharya, Crosignani, Eisert, and

[1]Another way to support lending to small and mid-sized firms in particular has been the Main Street Lending Program (MSLP) in the US, whereby banks continue to screen and originate loans that can then be sold to a special purpose vehicle maintained by the Fed (see, e.g., Minoiu, Zarutskie, and Zlate (2021) for an analysis of its effects).

[2]A growing number of papers analyze the role of banks and other lenders as conduits of public liquidity through government guaranteed loans to SMEs in Covid times both in Europe (e.g., Core and De Marco, 2020; Gonzalez-Uribe and Wang, 2020) and the US (e.g., Balyuk et al., 2020; Bartik, Cullen, Glaeser, Luca, Stanton and Sunderam, 2020; Cole, 2020; Duchin, Martin and Michaely, 2020; Granja, Makridis, Yannelis, and Zwick, 2020; Hubbard and Strain, 2020). The focus in these studies ranges from highlighting the importance of supply heterogeneity in the allocation of guaranteed loans to their implications on firm employment.

Eufinger, 2020; Laeven, Schepens, and Schnabel, 2020).

While perhaps valid, these arguments reflect a narrow view on bank lending decisions. As is well known, bank choices on the asset side also crucially depend on their capital structures. In particular, the degree of bank capitalization as well as the possibility for investors to withdraw their funds has been found to affect bank lending decisions (see, e.g., the evidence in Iyer and Puri, 2012; Iyer, Puri and Ryan, 2016; Martin, Puri and Ufier, 2018; Artavanis, Paravisini, Robles-Garcia, Seru and Tsoutsoura, 2019; or Carletti, De Marco, Ioannidou and Sette, 2020). In other words, the quality of a bank's assets, the threat of runs, and its capital structure are closely intertwined: investors react to signals on banks' fundamentals when deciding whether to withdraw their funds and, anticipating this, banks take investors' reactions into account when making their lending decisions. These considerations point to the need for evaluating the impact of PGSs on bank asset quality and, consequently, financial stability in a framework that incorporates the feedback effects between bank lending decisions and investors' behavior.

To tackle these issues, we present a model of financial fragility in the spirit of Goldstein and Pauzner (2005), which we enrich in two important dimensions. First, we assume that banks maximize profits, and fund themselves with equity in addition to demandable deposits. Second, we introduce a portfolio risk choice for banks by assuming that they can affect the success probability of loans when choosing their underwriting effort. These two aspects allow us to analyze the interaction between the asset and liability side of banks' balance sheet and to stress the importance of bank capital structure for the overall effects of the guarantees in terms of banks' underwriting and financial stability.

The model has two periods. Banks with some equity capital raise additional funds in the form of demandable debt and grant long-term loans to finance firms' projects. Depositors may leave their funds in the bank until projects mature in the second period or they may withdraw in the first period. If depositors wait until maturity, they obtain higher returns. However, the bank may become insolvent due to bad market fundamentals or poor underwriting standards. In such case, the bank incurs costs associated with bankruptcy, which further reduces the return to depositors.[3] When withdrawing early, by contrast, depositors obtain a return that depends on the capital

[3]Considerable empirical evidence shows that bank bankruptcy costs are substantial. For example, James (1991) finds that when banks are liquidated, bankruptcy costs are 30 cents on the dollar. The presence of bankruptcy costs contributes to the build up of social costs around bank failures and thus the need to reduce financial fragility. Our theoretical framework with endogenous bank portfolio choice and instability resulting from runs is well-suited to tackle this issue.

structure of the bank and the number of depositors requesting their money back. As is common in models of bank runs, depositors base their withdrawal decisions on a signal they receive in an intermediate period, as this provides information on the fundamentals as well as other depositors' behavior (see, e.g., Goldstein and Pauzner, 2005).

We first show that banks are subject to runs, whose probability decreases with the level of bank capitalization. In addition, banks with high levels of capital are subject to runs only when macroeconomic fundamentals are sufficiently poor (fundamental-driven runs), while banks with low capital are also prone to panic runs, meaning that their depositors may decide to run for reasons linked to strategic complementarity problems that arise when they anticipate other depositors may run. It follows that, for any level of capital, banks can only fail in the final period when their underwriting effort turns out to be unsuccessful.

We then turn to the analysis of the effects of loan guarantees on both the run probability and the bank's underwriting effort. We start by considering a scheme in which the government is in a first-loss position up to a pre-determined amount. In other words, whenever the borrower is unable to repay the promised amount to the bank, the government makes a transfer to the bank to cover part of the loss. Key for the analysis is the treatment of such transfer in case of bank default. We consider two cases. In the first one, the transfer by the government is subject to the same bankruptcy costs as any other bank asset. In the second one, the government guarantee is protected in case of bankruptcy and can be used to repay depositors. The two cases reflect different views on the nature of the bankruptcy costs. The case of full bankruptcy costs reflects a situation where bankruptcy losses originate primarily from inefficiencies in the bankruptcy procedures due to hold-up problems among creditors or inefficient judicial systems. By contrast, the case where the guarantee is protected in bankruptcy captures a setting where bankruptcy losses stem primarily from the illiquidity of bank assets, such as loans, and hence do not apply to more liquid assets such as governmental transfers.

In the case with full bankruptcy costs, the introduction of loan guarantees reduces depositors' run probability and improves bank underwriting effort. As with any form of insurance, the guarantee increases the range in which the bank is able to make the promised repayment to depositors in the final date, thus reducing their incentives to withdraw prematurely. However, the introduction of the loan guarantee has multiple effects on the bank's monitoring effort. First, there is a direct beneficial effect as banks obtain higher profits when there is no run and they are solvent. Second, the reduction in the run probability increases the likelihood the bank will be able to enjoy its second period return. Both of these effects improve bank underwriting incentives. This result contrasts with perceived wisdom related to other forms of guarantees, such as deposit insurance, where their use is often posited to lead to reduced monitoring and more risk-taking.

We then turn to the case where the loan guarantee is bankruptcy-protected. While, similar to before, the run probability decreases, the bank's underwriting effort instead decreases for poorly capitalized banks. The reason is that the introduction of the loan guarantee now lowers the sensitivity of the run probability to changes in underwriting standards, thus reducing the benefit for the bank to exert effort to control risk-taking. The latter effect is dominant when the bank has low capital and, as a consequence, faces a high run risk, so that the loan guarantee becomes bad for bank incentives. For more capitalized banks, the previous results hold and underwriting standards improve in the presence of the loan guarantee.

We then extend the analysis to consider another type of loan guarantee where there is sharing of losses between the government and the bank.[4] We find that all results concerning the run probability and bank underwriting efforts remain qualitatively the same. However, the two schemes differ in terms of costs and effectiveness for bank incentives. For a given run probability, the first-loss guarantee provides greater incentives to the bank but at higher costs.

One crucial element of the analysis is whether the guarantee accrues to the bank conditional on its ability to control risk. In the model, the guarantee is disbursed whenever the firm is unable to repay the bank. However, whether the bank or its creditors benefit from the guarantee when the bank's underwriting effort is successful (i.e., when positive project returns are realized) depends on the treatment of the guarantee in bankruptcy. In the case of full bankruptcy costs, both the bank and depositors only stand to receive anything if there is no run and the bank remains solvent. By contrast, in the case of bankruptcy-protected guarantees, depositors also receive some payment in the final period when the bank's underwriting effort is unsuccessful and the bank defaults. This reduces the sensitivity of depositors' incentives to run to the bank's underwriting standards, thus indirectly benefiting the bank and reducing its incentives. Therefore, the treatment of the guarantee in bankruptcy becomes de facto equivalent to a conditionality assumption.

In a further step, we extend the analysis to include banks' project continuation decisions by

[4]The two schemes we consider mirror the structures of the guarantees used in practice in addressing the need for sustaining lending in the aftermath of the Covid-19 pandemic (see, for example, European Commission, 2020).

allowing them to liquidate projects at the interim date. In the absence of runs, all banks would engage in *evergreening* (i.e., continue projects that would be efficient to liquidate) and, in line with the empirical evidence (see e.g., Blattner, Farinha and Rebelo, 2021; and Schivardi, Sette and Tabellini, 2021), the more so the lower is their level of capitalization. Once runs are taken into account, loans can be liquidated early either because of runs or directly by the bank. For banks with low capital, depositors exerts a strong disciplinary force and projects get liquidated early because of depositor runs. By contrast, when banks have high capital, depositors are more passive and early liquidation occurs primarily as a result of banks' decisions. In this context, the introduction of loan guarantees leads to more evergreening since depositors' incentives to run decrease, while banks' incentives to continue inefficient projects increase, in particular for worse-capitalized banks. This result is in line with the evidence in Dursun-de Neef and Schandlbauer (2021) that worse-capitalized banks showed a lower increase in their delinquent loans and loan restructuring during the pandemic relative to better-capitalized competitors.

Our main contribution is to analyze the role of public loan guarantees, such as those introduced in the recent pandemic, in a framework where both banks' portfolio risk choices and financial fragility are derived endogenously. Guarantees on lending contracts are not uncommon in practice and have been studied in prior literature. Evidence in Beyhaghi (2021) shows that over one-third of corporate loans issued by US banks are guaranteed by separate legal entities, mostly in the form of personal or corporate guarantors. Similarly, Ahnert and Kuncl (2021) report that 62% of outstanding residential mortgages were insured by the US government through the Government Sponsored Enterprises in 2018. Ahnert and Kuncl (2021) find that this type of third-party loan guarantee decreases lending standards, but improves market liquidity. In their model, lenders have the possibility to pass default risk to an outside guarantor upon origination, thus avoiding costly screening. We also analyze loan guarantees upon origination, but in a context where these are not an alternative to bank screening.

Our paper is related to the literature studying alternative ways to transfer credit risk onto third parties after loan origination. For example, Parlour and Winton (2013) study the effects of credit default swaps (CDSs) on banks' monitoring incentives as an alternative to loan sales in secondary markets. In a repeated setting where reputational considerations matter, they show that CDSs tend to dominate loan sales only for riskier credits, while their effects on bank monitoring depend on credit quality. In contrast, we focus on loan guarantees where banks retain both cash flow and control rights, and show that in the presence of these guarantees bank incentives depend on the level of capitalization as well as on the nature of bankruptcy costs.

A large strand of literature has focused on the role of government guarantees such as deposit insurance or other forms of implicit guarantees on banks' liabilities. On the one hand, government guarantees are thought to have a positive role in preventing panic among investors, and hence help stabilize the financial system (e.g., Diamond and Dybvig, 1983). On the other hand, they may distort banks' incentives, leading to an increase in financial fragility (see, e.g., Calomiris, 1990, and Acharya and Mora, 2015). Reconciling the two views, more recent studies show that government guarantees can be welfare improving because they induce banks to improve liquidity provision (Keister, 2016), although in a way that sometimes increases the likelihood of runs or creates distortions in banks' behavior (Allen, Carletti, Goldstein and Leonello, 2018). In this paper, we focus on PGSs for loans rather than deposits and study how they affect bank behavior and financial stability through their interaction on the asset side of the balance sheet. The idea that a government guarantee on deposits can actually be good for incentives has been studied in Cordella, Dell'Ariccia, and Marquez (2018), who show that, by reducing a bank's cost of funding, a deposit guarantee increases the return to the bank and creates greater incentives to monitor.

Another strand of literature has instead analyzed credit risk in the form of bank monitoring effort and the role of bank capital, but generally without including financial fragility. For example, Holmstrom and Tirole (1997) study the incentive problem for a bank to monitor a borrower and show how this incentive depends on the amount of capital the bank has. Hellmann, Murdock, and Stiglitz (2000), Repullo (2004), Morrison and White (2005), Dell'Ariccia and Marquez (2006), Allen, Carletti, and Marquez (2011), Mehran and Thakor (2011), and Dell'Ariccia, Laeven, and Marquez (2014) study settings where banks are subject to moral hazard in their monitoring decisions, and where equity capital can serve as a way to improve bank incentives (see also Thakor, 2014, for a survey). As such, banks may have incentives to raise capital even in the absence of capital requirements (e.g., Allen, Carletti, and Marquez, 2011). None of these papers, however, studies how bank monitoring is affected by, and in turn affects, financial fragility in the form of bank runs. An exception is Kashyap, Tsomocos, and Vardoulakis (2019), who focus on the effect of capital and liquidity for credit and run risk. Instead, we are interested in the effects of loan guarantees for the banks' monitoring choice and the likelihood of runs.

Our analysis of the effect of loan guarantees on banks' incentives to engage in evergreening connects to a recent literature on zombie lending, or in other words the provision of credit to firms already in distress.[5] In Hu and Varas (2021), evergreening emerges as a result of the existence of dynamic lending relationships and the advantages that a relationship bank can obtain from helping its borrowers to have a strong reputation. Bruche and Llobet (2014) show that zombie lending arises from bank limited liability, and that a regulatory intervention in the form of asset buybacks or subsidizing the foreclosure of bad loans may be effective in reducing banks' incentives to engage in evergreening. Relative to these papers, we focus on the effect that the introduction of loan guarantees has on bank incentives to provide credit to firms in distress and highlight the role of bank capital. Related to this last point, Blattner et al. (2021) show empirically that, following the introduction of more stringent capital requirements in Portugal, weak banks started to provide credit to distressed firms for which the bank had been underreporting loan loss provisions prior the regulatory change. A similar result is also found in Schivardi et al. (2021), who show that during the 2008 financial crisis undercapitalized banks were more likely to provide credit to zombie firms than better capitalized ones. In line with this, in our framework poorly capitalized banks have the greatest incentive to engage in evergreening. In equilibrium, depositor discipline in terms of runs interact with the bank liquidation incentives in determining the extent of evergreening.

The important role that depositors' discipline plays on banks through the use of demandable contracts and the associated threat of a run has been highlighted in Calomiris and Kahn (1991). In their paper, banks can abscond with the realized returns and depositors discipline banks by having the possibility to withdraw their funds prematurely. Similarly, in our framework the run threshold decreases with underwriting standards and the bank takes this into account when choosing whether and how monitor its loans.

The paper proceeds as follows. Section 2 presents the baseline model. Section 3 characterizes the equilibrium without guarantees in terms of both bank monitoring effort and financial fragility. Section 4 introduces public guarantees, distinguishing between first loss and loss absorption schemes. For either scheme, we analyze both the case when all bank revenues are lost in bankruptcy and when the guarantee is bankruptcy-protected. Section 5 studies the introduction of deposit insurance and compares it with loan guarantees. Section 6 analyzes the effect of introducing loan guarantees for banks' incentives to engage in evergreening, while Section 7 contains concluding remarks. All proofs can be found in the Appendix.

[5] A number of earlier contributions focused on the Japanese experience; see, e.g., Peek and Rosengren (2005); and Caballero, Hoshi and Kashyap (2008).

2 The model

Consider a three date economy ($t = 0, 1, 2$) with banks and a large number of atomistic risk-neutral investors who have a unitary endowment at date 0.[6] Each bank has access to a risky loan requiring one unit of investment. Each bank has (internal) capital of k, and, at date 0, raises the remainder $1-k$ from investors in the form of demandable debt, as described further below. We normalize depositors' outside option to 1, so that the expected payoff they must receive for depositing their endowment at a bank must be at least equal to this outside option.

Each bank can make loans to finance a firm's risky project. The firm's project, if held to maturity (i.e., until $t = 2$), yields a return $\widetilde{P}$, with

$$\widetilde{P} = \begin{cases} R\theta & \text{w.p. } q \\ 0 & \text{w.p. } 1-q \end{cases} .$$

The date 2 return on the project depends on the fundamental of the economy θ, with $\theta \sim U[0, 2]$, and $q \in [0, 1]$, which capture the level of macroeconomic risk and the amount of underwriting or monitoring effort chosen by a bank, respectively.[7] Choosing a higher probability of success q is costly, and we assume that the bank bears a private non-pecuniary cost of $c\frac{q^2}{2}$. If liquidated early, at $t = 1$, the project yields a (fixed) liquidation value $L \leq 1$. We normalize the interest rate a bank receives on its loan to R, so that for $\theta \geq 1$, the bank receives full repayment, while for $\theta < 1$, there is partial default, with the bank receiving $R\theta$ and suffering losses $R(1-\theta)$.[8] Furthermore, we assume $\frac{1}{2}\int_0^1 qR\theta d\theta + \frac{1}{2}\int_1^2 qRd\theta - c\frac{q^2}{2} = \frac{3}{4}qR - c\frac{q^2}{2} > 1$ for some q, so that granting loans to finance firms' projects dominates storing as long as the bank chooses a sufficiently high monitoring effort.

The mass $1-k$ of investors at each bank holds a standard demandable deposit contract. At date 1, a depositor can redeem its deposit from the bank at par, i.e., for the same amount that was originally deposited, while he receives $r_2 > 1$ at date 2 if he waits until then.[9] The promised

[6] The exact number of banks is immaterial as long as there are relatively more investors than banks, so that investors need only have their reservation utility satisfied in order to be willing to deposit at a bank. Since each bank has access to a single project, we abstract from additional issues that may emerge as a result of competitive interaction across banks when competing to grant loans.

[7] We will use the terms "underwriting" and "monitoring" interchangeably throughout the paper to represent the effort choices banks must make to improve the repayment probability of the loans they extend.

[8] Note that this implies the borrower, who owns the project, receives a return of $R(\theta-1)$ when $\theta > 1$ and the project succeeds (with probability q), and 0 otherwise, when $\theta \leq 1$ or with probability $(1-q)$. In other words, loan default occurs when fundamentals, as measured by θ, are low and with probability $1-q$.

[9] We consider debt to be demandable, which is equivalent to assuming long-term debt with the possibility for investors to withdraw earlier for a positive repayment. The optimality of demandable debt has been justified in

repayments $\{1, r_2\}$ are paid as long as the bank has enough resources. If depositors choose to withdraw at date 1, the bank liquidates as much of its assets as needed to satisfy withdrawals, obtaining $L < 1$ per unit liquidated, and carrying to time 2 any remaining amount. If the bank has insufficient resources to meet depositors' demands at date 1, all its assets are liquidated and the $1-k$ depositors receive a pro-rata share of the liquidation value L.[10] By contrast, if the bank fails to repay depositors r_2 at date 2, the bank enters a bankruptcy procedure and depositors experience losses as a result. For simplicity, we assume bankruptcy costs are 100%, so that depositors receive nothing upon insolvency of the bank at date 2. The bankruptcy costs may originate either from coordination failures among a bank's creditors, which makes it difficult and costly for them to seize the remaining value of the bank, or from the illiquidity of the bank's assets, where some value is lost when selling them to alternative users/lenders. The different possible sources of bankruptcy costs will play an important role in the analysis of the loan guarantee scheme, as we discuss in detail below.

The state of the economy θ is realized at the beginning of date 1, but is publicly revealed only at date 2. After θ is realized at date 1, each depositor receives a private signal s_i of the form

$$s_i = \theta + \varepsilon_i, \tag{1}$$

where ε_i are small error terms that are independently and uniformly distributed over $[-\varepsilon, +\varepsilon]$. After the signal is realized, depositors decide whether to withdraw at date 1 or wait until date 2.

The timing of the model is as follows. At date 0, banks raise deposits with a deposit contract $\{1, r_2\}$, and then choose the riskiness of their portfolios q. At date 1, after receiving the private signal about the state of the fundamentals θ, depositors decide whether to withdraw early or wait until date 2. At date 2, the bank's portfolio return is realized and depositors that chose to wait are repaid.

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the literature by the presence of asymmetric information problems in credit markets (see, e.g., Flannery, 1986; and Diamond, 1991), conflicts between bank managers and shareholders (see e.g., Calomiris and Kahn, 1991; Diamond and Rajan, 2001; and Eisenbach, 2017) and idiosyncratic liquidity shocks to banks' depositors (e.g., Diamond and Dybvig, 1983). As we argue in Section 4, assuming an early repayment equal to 1 is without loss of generality in terms of a bank's exposure to runs.

[10]Following Goldstein and Pauzner (2005) and related papers, we assume that there are no bankruptcy costs at date 1, only the $1-L$ units of resources that are lost due to the premature liquidation of banks' loans. Assuming costs stemming from bankruptcy at date 1 does not qualitatively affect our results. Calculations can be provided to interested readers.

3 Economy without guarantees

In this section, we characterize the allocation for a baseline case where there are no guarantees. We start by analyzing depositors' withdrawal decisions at date 1, taking the deposit contract $\{1, r_2\}$ and the riskiness of the portfolio q as given. Then, we move on to the choice of the portfolio riskiness q and the terms of the deposit contract r_2.

3.1 Depositors' withdrawal decision

Depositors base their withdrawal decisions on the signal they receive, as the signal gives them information about the economy's fundamentals θ and the actions of all other depositors in the bank. When he receives a high signal, a depositor expects the return of his bank's loan portfolio to be high and, at the same time, he expects that other depositors have also received a high signal. This lowers his incentives to withdraw early (i.e., run). Conversely, when a depositor receives a low signal, he expects a low return for the bank, and hence less cash available to repay depositors. He also expects a large number of depositors to run. As a result, he has a higher incentive to run. This suggests that depositors withdraw at date 1 when the signal is low enough, and wait until date 2 when the signal is sufficiently high.

To show this formally, we first examine two regions of extremely bad and extremely good fundamentals, where each depositor's action is based on the realization of the fundamentals θ irrespective of his beliefs about other depositors' behavior. We start with the lower region.

Lower Dominance Region. The lower dominance region of θ corresponds to the range $[0, \underline{\theta})$ in which running is a dominant strategy. Upon receiving a signal in this region, a depositor is certain that the date 2 expected repayment is lower than the payment from withdrawing at date 1, even if no other depositors were to withdraw. Given the presence of bankruptcy costs, the depositor knows that at date 2 he will receive $qr_2 > 1$ only if the bank is solvent and thus able to make the promised repayment r_2 to all $1-k$ depositors, and 0 otherwise. Thus, his incentives to run depend on whether the bank is solvent or not, which boils down to the threshold $\underline{\theta}(k)$ being the solution to

$$R\theta = (1-k)\, r_2. \tag{2}$$

Upper Dominance Region. The upper dominance region of θ corresponds to the range $[\overline{\theta}, 2]$ in which fundamentals are so good that waiting to withdraw at date 2 is a dominant strategy. Following Goldstein and Pauzner (2005), we construct this region by assuming that in the range

$[\overline{\theta}, 2]$ the bank's loan portfolio can be liquidated at date 1 at a value $\overline{L} \geq 1$. The higher liquidation value guarantees that a bank needs to liquidate no more than 1 unit of its investment for each withdrawing depositor. As a result, each deposited unit yields to the bank a return $q \min\{R\theta, R\}$ at date 2. This implies that, for any $\theta \geq 1$, a depositor waiting until date 2 receives qr_2, which must be greater than 1 in order for intermediation to be feasible.[11] Then, for simplicity, by setting $\overline{\theta} = 1$, we ensure that waiting until date 2 is a dominant strategy for any $\theta \geq 1$.

The Intermediate Region. When the signal indicates that θ is in the intermediate range, $[\underline{\theta}, \overline{\theta})$, a depositor's decision to withdraw early depends on the realization of θ as well as on his beliefs regarding other depositors' actions. To see how, we first calculate a depositor's utility differential between withdrawing at date 2 and at date 1. Using n to represent the fraction of depositors who choose to withdraw early, this differential is given by

$$v(\theta, n) = \begin{cases} qr_2 - 1 & \text{if } 0 \leq n \leq \widehat{n}(\theta) \\ 0 - 1 & \text{if } \widehat{n}(\theta) \leq n \leq \overline{n} \\ 0 - \frac{L}{(1-k)n} & \text{if } \overline{n} \leq n \leq 1 \end{cases},$$

where $\widehat{n}(\theta)$ solves

$$R\theta\left(1 - \frac{n(1-k)}{L}\right) - (1-n)(1-k)r_2 = 0, \tag{3}$$

while $\overline{n}$ solves

$$L = n(1-k).$$

The threshold $\widehat{n}(\theta)$ represents the proportion of depositors running at which a bank is no longer able to repay r_2 to those waiting until date 2, while $\overline{n}$ captures the number of withdrawing depositors at which a bank liquidates the entire portfolio at date 1. As shown in Figure 1 and Figure 2, the shape of the function $v(\theta, n)$ depends on whether $1 - k \gtreqless L$, that is, on whether the amount of deposits is greater or smaller than the liquidation value L. The case when $1 - k \leq L$ is illustrated in Figure 1, which shows that the function $v(\theta, n)$ is constant in n, the fraction of depositors withdrawing early, and equal to $qr_2 - 1 > 0$ if $\theta \geq \underline{\theta}$ and $0 - 1 < 0$ if $\theta < \underline{\theta}$. Hence, when $1 - k \leq L$, $v(\theta, n)$ is either positive or negative depending on whether θ is above or below $\underline{\theta}$, which implies that a depositor's incentive to run is independent of what others do. This occurs because, in this case, a bank has enough resources at date 1 to repay all withdrawing depositors and still make the

[11] If $qr_2 < 1$, depositors would never find it optimal to wait until date 2 and would strictly prefer to withdraw early, at date 1. Anticipating this, all depositors would prefer to pursue whatever alternative investment is available to them yielding 1 rather than deposit at the bank. Hence, a minimum requirement for intermediation to be feasible is that the bank chooses a high enough level of monitoring so that, given the equilibrium r_2, $qr_2 > 1$. This can readily be achieved for c sufficiently low and/or R sufficiently high.

promised repayment r_2 to those waiting until the final date. It follows that runs are only triggered by the fear that low fundamentals lead to bank insolvency at the final date.

Insert Figure 1

The case when $1 - k > L$ is illustrated in Figure 2. In this case, the function $v(\theta, n)$ is a constant and positive for $0 \leq n \leq \widehat{n}(\theta)$, while in the range $\widehat{n}(\theta) \leq n \leq \overline{n}$ it is always below zero.

Insert Figure 2

When $1 - k > L$, there are more depositors at the bank that may run at date 1 and thus the bank may be forced to prematurely liquidate a large amount of assets if many of them demand their funds before maturity. This introduces strategic complementarities in depositors' withdrawal decisions, as is typical in models of runs (e.g., Goldstein and Pauzner, 2005): the expected payoff of depositors waiting until date 2 is decreasing in the proportion n of depositors withdrawing at date 1, so that their incentive to run increases with n. In other words, a depositor's withdrawal decision depends on the others depositors' behavior and runs are driven by fears of large withdrawals in the form of panics.

Throughout, we focus our results on the limiting case where $\varepsilon \to 0$, so that the noise in depositors' information becomes vanishingly small. This implies that all depositors behave alike: they all either withdraw at date 1, thus originating a run, or wait until date 2. The following proposition characterizes depositors' withdrawal decisions and the threshold value of fundamentals θ below which runs occur.

Proposition 1 *The run risk depends on the level of bank capitalization as follows:*

a) When $1 - k \leq L$, runs are triggered only by bank insolvency at date 2 and they occur when θ falls below the threshold $\underline{\theta}(k)$, where

$$\underline{\theta}(k) = \frac{(1-k)\, r_2}{R}, \tag{4}$$

with $\underline{\theta}(k)$ being decreasing in k: $\frac{\partial \underline{\theta}(k)}{\partial k} < 0$.

b) When $1-k > L$, runs are driven also by panics and they occur when θ falls below the threshold $\theta^*(q, k)$, where

$$\theta^*(q,k) = \underline{\theta}\frac{q r_2 - \pi_1}{q r_2 - \pi_1 \frac{(1-k)}{L}}, \tag{5}$$

where $\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n} dn$*. The threshold* $\theta^*(q,k) \in (\underline{\theta}(k), 1)$ *decreases with* q*,* L *and* k*:* $\frac{\partial \theta^*(q,k)}{\partial q} < 0$*,* $\frac{\partial \theta^*(q,k)}{\partial L} < 0$*, and* $\frac{\partial \theta^*(q,k)}{\partial k} < 0$*.*

The proposition shows the importance of bank capitalization for run risk. When a bank is well capitalized (i.e., $1-k \leq L$) runs are only driven by poor fundamentals, and the critical threshold $\underline{\theta}$ is decreasing in the amount of capital k. The reason is that when $1-k \leq L$, the bank has enough resources to repay all withdrawing depositors in full at date 1, while still having resources left at date 2 to repay any depositor waiting until then. In contrast, when a bank has little capital, it is exposed to runs over a larger range of fundamentals (i.e., for $\theta < \theta^*$ with $\theta^* > \underline{\theta}$) due to the presence of strategic complementarities. The panic run threshold θ^* decreases with the level of capitalization k, but is also decreasing with the safety of the portfolio as measured by q and the liquidation value L. In particular, increasing the monitoring effort q reduces the bank's exposure to panic runs (i.e., $\frac{\partial \theta^*}{\partial q} < 0$) because a higher q increases depositors' expected payoff from waiting until date 2. Regarding the liquidation value L, the reduction of the run threshold comes from the fact that a higher L reduces the bank's liquidation needs, thus leaving more resources for depositors withdrawing at the final date.

In order to focus on cases where run risk is potentially severe, at least for very poorly capitalized banks, we assume that there exists a value of capital $\widehat{k} \in [0, 1-L)$ such that $\theta^* \to 1$ as $k \to \widehat{k}$ at the equilibrium. In other words, run risk is maximal for banks with sufficiently low levels of capital (the threshold level of capital $\widehat{k}$ can be arbitrarily small).[12] The majority of our results do not depend on this assumption and are robust to lower levels of run risk, and we explicitly highlight below which result makes use of the assumption on maximal run risk.

3.2 Bank's date 0 decisions

Having characterized depositors' withdrawal decisions, we now solve for banks' underwriting standards q as well as the repayment r_2. Since we want to consider all possible levels of capitalization k, in what follows, we use θ^R to denote the relevant run threshold, i.e., $\theta^R = \underline{\theta}$ when $1-k \leq L$ and $\theta^R = \theta^*$ when $1-k > L$.

For a given level of capital k, each bank chooses its underwriting effort q and depositors' repayment r_2 in order to maximize its expected profits, anticipating depositors' withdrawal decisions

[12] It is not difficult to construct explicit numerical examples showing that for many choices of the parameters R, c, and L, the equilibrium run probability θ^* satisfies this condition for low enough k, so that the assumption is not vacuous. Details provided upon request.

at date 1. The problem for a bank is

$$\max_{q,r_2} \Pi = \frac{1}{2}\int_0^{\theta^R} q \max\left\{R\theta\left(1-\frac{1-k}{L}\right),0\right\} d\theta + \frac{1}{2}\int_{\theta^R}^1 q\left[R\theta-(1-k)\,r_2\right] d\theta + \frac{1}{2}\int_1^2 q\left[R-(1-k)\,r_2\right] d\theta - \frac{cq^2}{2} \tag{6}$$

subject to

$$\underbrace{\tfrac{1}{2}\int_0^{\theta^R} \min\left\{\frac{L}{1-k},1\right\} d\theta}_{\text{utility obtained in a run}} + \tfrac{1}{2}\underbrace{\int_{\theta^R}^2 qr_2 d\theta}_{\text{utility obtained if no runs occur}} \geq \underbrace{1}_{\text{outside investment opportunity}}, \tag{7}$$

and

$$\Pi \geq k. \tag{8}$$

The first three terms in (6) capture the three instances when the bank accrues positive profits at date 2. First, when $1-k \leq L$ so that $\theta^R = \underline{\theta}$ and a run occurs (i.e., for $\theta < \underline{\theta}$), the bank does not have to liquidate its entire portfolio at date 1 to satisfy the early withdrawals, and thus it obtains the return $R\theta$ on the $1-\frac{1-k}{L}$ units of assets that are carried over until date 2 if the project is successful (i.e., with probability q). By contrast, when $1-k > L$, the bank liquidates the entire portfolio at date 1 and makes zero profits when a run occurs. Second, for any level of k, when no runs occur and $\theta \in (\theta^R, 1]$, the profits are given by the return $R\theta$ minus the repayment to depositors, $(1-k)\,r_2$. Finally, for any level of k, when no runs occur and $\theta \in [1,2]$, the bank obtains the fixed return R net of depositors' repayments. The last term in (6) represents the monitoring cost $\frac{cq^2}{2}$ the bank bears.

The condition in (7) represents depositors' participation constraint and states that the expected repayment from depositing in a bank is not lower than the value of the outside investment opportunity. By depositing in a bank, depositors expect to receive the minimum between the pro-rata share $\frac{L}{1-k}$ and the promised repayment 1 if there is a run (i.e., when $\theta \leq \theta^R$), and qr_2 if there is no run (i.e., $\theta > \theta^R$). From (7), one can see that a minimal condition for depositors to provide funds to the bank is $qr_2 \geq 1$. This is the case since the payoff in the event of a run is $\min\left\{\frac{L}{1-k},1\right\} \leq 1$. Finally, the inequality in (8) is simply a non-negativity constraint on the profits. We have the following result.

Proposition 2 *Banks choose the monitoring effort q and the depositors' repayment r_2 as follows:*

a) When $1-k \leq L$, each bank chooses $\underline{q}$ as a solution to

$$\frac{1}{2}\int_0^{\underline{\theta}} R\theta\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}}^{1}\left[R\theta-(1-k)\,r_2\right]d\theta+\frac{1}{2}\int_1^2\left[R-(1-k)\,r_2\right]d\theta-cq=0, \quad (9)$$

where $r_2>1$ is the solution to (7) holding with equality;

b) When $1-k>L$, each bank chooses q^* as a solution to

$$\frac{1}{2}\int_{\theta^*}^{1}\left[R\theta-(1-k)\,r_2\right]d\theta+\frac{1}{2}\int_1^2\left[R-(1-k)\,r_2\right]d\theta-\frac{1}{2}\frac{\partial\theta^*}{\partial q}q\left[R\theta^R-(1-k)\,r_2\right]-cq=0, \quad (10)$$

where $r_2>1$ is the solution to

$$-\frac{1}{2}\frac{\partial\theta^*}{\partial r_2}\left[R\theta^*-(1-k)\,r_2\right]-\frac{1}{2}\int_{\theta^*}^{2}(1-k)\,d\theta=0 \quad (11)$$

when $\mu=0$, and to (7) holding with equality when $\mu>0$, where μ is the Lagrange multiplier on depositors' participation constraint as defined in the Appendix.

In choosing q, a bank trades off the marginal cost of an increase in q with its marginal benefit. The former is given by cq. The latter, which is captured by the first three terms in either (9) or (10), is given by the increase in expected profits resulting from a higher probability of success in all the instances where the bank obtains positive profits, which depends on the level of k. When $1-k>L$, there is an additional term reflecting how an increase in q affects the run threshold. This is captured by the third term in (10) since by monitoring more the bank also reduces the risk of a run, i.e., $\frac{\partial\theta^*}{\partial q}<0$. This provides an additional incentive for the bank to choose a high q, separately from the fact that a higher q increases the probability the bank's loan is repaid.

Proposition 2 shows that also the determination of r_2 depends on the level of capitalization of the bank. When a bank is well capitalized so that $1-k\leq L$, it finds it optimal to choose the lowest possible repayment r_2 consistent with depositors being willing to provide funds to the bank. This is the case because the run threshold $\underline{\theta}$ increases with r_2 and so bank's profits are strictly decreasing in r_2. By contrast, when $1-k>L$ and panic runs may occur, in choosing the optimal repayment r_2 a bank also accounts for the potentially beneficial effect that a higher r_2 has on the run threshold θ^*, since θ^* is decreasing in r_2. As a result, a bank may find it optimal to choose a repayment r_2 which leaves depositors' participation constraint (7) slack. In other words, in some cases, it may be optimal for a bank to leave rents to depositors in order to reduce its exposure to runs and ultimately increase expected profits.

4 Public loan guarantee schemes

In this section, we study how the introduction of loan guarantees affects bank risk-taking and overall bank stability. While the specific terms may differ across countries, PGSs take essentially one of two forms: first-loss or loss-sharing.[13] In the former, losses are first attributed to the State up to a certain limit, and only then to the credit intermediary. In the latter, by contrast, losses are sustained proportionally by the credit institutions and the State in some pre-determined proportions. We begin by focusing on a first-loss loan guarantee and show later in Section 4.3 that the main insights of the analysis carry over to the loss-sharing scheme.

In the context of our theoretical framework, the rationale for the introduction of the guarantee stems from an unexpected negative shock such that, absent some form of stimulative policy, banks may opt not to lend, thus worsening the real effect of the shock. A simple example would be a shock to the liquidation value of projects, L, causing it to fall and making loans less attractive to banks.[14] Another possibility would be a reduction in R, the expected return of the projects.[15] For the sake of simplicity, we leave this part out of the formal development of the model. Moreover, since the guarantees are introduced as responses to unanticipated shocks, we assume they are put in place after banks have raised financing but before making their bank lending decisions.

For each guarantee scheme, we will consider two cases concerning the treatment of the loan guarantee in case the bank is insolvent at date 2. In the first case, we will assume that the amount provided by the government is lost in bankruptcy in the same way as the return of any other asset the bank carries until date 2. In the second case, by contrast, we will assume that the guaranteed amount is protected from dissipative costs in bankruptcy and, thus, the transfer from the government to a bank can be used to repay investors. We will refer to the first case as "full bankruptcy costs" and to the second as "bankruptcy protected" to indicate that the guarantee is not subject to losses arising during bankruptcy. The first case captures the idea that the bankruptcy costs primarily originate from inefficiencies in the bankruptcy procedures due to hold-up problems

[13] See, for example, European Commission (2020) for a description of the loan guarantee schemes used in Europe during the pandemic.

[14] Interestingly, an unexpected shock to the liquidation value also rationalizes the normalization of the date 1 interest rate to 1, since for any value of the date 1 interest rate there is always a negative shock to L sufficiently large that panic runs are possible.

[15] Note that even though banks may find lending unattractive, socially the loan may still represent a positive net present value (NPV) investment, both because banks do not capture the full surplus of the loan, and because the run risk lowers the value to the banks. Policies that reduce run risk (e.g., deposit insurance or suspension of convertibility) would make lending more attractive, even if the loan itself was not subsidized.

among creditors or inefficient judicial systems and, as a result, resources are lost if the bank defaults at date 2. The second case would be consistent with a setting where bankruptcy costs primarily stem from illiquidity associated with selling assets. The guarantee paid by the government would likely be in cash or other such liquid assets and less subject to dissipation. As seen below, the distinction between these two cases turns out to be important for how guarantees affect bank incentives.

Finally, in order to isolate the effect of loan guarantees, we introduce them into a setting where there are no other guarantees already in place. In practice, of course, loan guarantees, being used as stimulative policy tools, are typically layered on top of other existing guarantees, such as deposit insurance, which we discuss later in Section 5. In our framework, the two types of guarantees – loan and deposit – are independent of each other and in fact turn out to have no complementary effects on bank incentives. Hence, the findings presented below are largely unchanged if the loan guarantees come on top of already-present deposit guarantees.

4.1 First-loss guarantee scheme with full bankruptcy costs

We start by considering the case where the government is in a first-loss position: it guarantees any losses up to an amount Rx, with any losses in excess of this amount being borne by the bank. This means that the government will transfer an amount $R\min\{x, 1-\theta\}$ to the bank when the borrower is unable to repay the promised amount R, so that for any $\theta \geq 1-x$ the bank suffers no losses and receives the full payment R, whereas for $\theta < 1-x$ the bank's return declines as θ falls since the losses are greater than the guarantee provided. Figure 3 describes how the introduction of the guarantee modifies the bank's payoff as a function of the fundamental θ. As shown in the figure, the bank now obtains the full repayment R for $\theta \in [1-x, 1]$ and a greater payoff $R(\theta + x)$ in the region with partial default, i.e., for $\theta \in [0, 1-x]$.

Insert Figure 3

As in the case without the loan guarantee, we start by characterizing depositors' withdrawal decisions and then move on the choice of q by banks. Consider for now that the transfer is lost in bankruptcy. Since our objective is to study how guarantees affect bank incentives to control risk-taking, in what follows we will sometimes divide the loan guarantee x into two parts, $x_1 < \frac{1-k}{R}$ and $x_2 \leq x_1$, corresponding to payments made when the bank's monitoring effort pays off and the project succeeds, with probability q, and when this effort does not pay off and the loan portfolio fails to produce anything, which occurs with probability $1-q$, respectively. In practice, of course, distinguishing between such cases may not be easy or even feasible. Separating the payment into two components allows us to tease out precisely how bank incentives are affected by the introduction of the loan guarantee, even if any guarantee is in reality likely to cover both possibilities, at least partly. The condition $\frac{1-k}{R} > x_1 \geq x_2$ ensures that the government's transfer alone is not sufficient to guarantee that depositors can always withdraw early at par.

The following proposition characterizes depositors' withdrawal decisions. We use the subscript x to indicate the case of the first-loss guarantee of size x, with full bankruptcy costs.

Proposition 3 *With a first-loss guarantee $x = (x_1, x_2)$ and full bankruptcy costs, runs occur for $\theta < \theta_x^R < \theta^R$ as given by*

$$\theta_x^R = \theta^R - x_1, \tag{12}$$

where $\theta_x^R = \underline{\theta}_x$ and $\theta^R = \underline{\theta}$ for $1-k \leq L$, while $\theta_x^R = \theta_x^*$ and $\theta^R = \theta^*$ for $1-k > L$. In either case, the threshold θ_x^R decreases with x_1, k, L and q, while it is independent of x_2: $\frac{\partial \theta_x^R}{\partial x_1} = -1 < 0$, $\frac{\partial \theta_x^R}{\partial k} < 0$, $\frac{\partial \theta_x^R}{\partial L} < 0$, $\frac{\partial \theta_x^R}{\partial q} < 0$ and $\frac{\partial \theta_x^R}{\partial x_2} = 0$.

As in the case without a loan guarantee, runs can be only fundamental-driven (when $1-k \leq L$) or also panic-driven (when $1-k > L$), but the probability of runs is now reduced for any given level of bank capitalization. Moreover, k, L and q have the same effect on the run threshold θ_x^R as in the economy without guarantees. As shown in Figure 4, the threshold θ_x^R decreases with the amount guaranteed because, ceteris paribus, x_1 increases the range in which the bank is able to make the promised repayment to depositors at date 2, thus reducing their incentives to withdraw prematurely. Importantly, the run threshold θ_x^R only depends on the amount x_1 transferred in the case the bank's monitoring effort pays off and, with probability q, the project succeeds, but not on the transfer x_2 when instead the project fails, with probability $1-q$. The reason is that the amount x_2 does not impact depositors' payoff at date 2 as x_2 is lost in bankruptcy.

Insert Figure 4

Having characterized the run thresholds, we now move on to the choice of q. Each bank's optimization problem is as follows:

$$\max_{q} \frac{1}{2}\int_{0}^{\theta_x^R} q \max\left\{R\left(\theta+x_1\right)\left(1-\frac{(1-k)}{L}\right),0\right\} d\theta+\frac{1}{2}\int_{\theta_x^R}^{1-x_1} q\left[R\left(\theta+x_1\right)-(1-k)\, r_2\right] d\theta \quad (13)$$
$$+\frac{1}{2}\int_{1-x_1}^{2} q\left[R-(1-k)\, r_2\right] d\theta-\frac{cq^2}{2},$$

where r_2 is characterized in Proposition 2 since the guarantee scheme is assumed to be an unanticipated shock to the banking sector, and θ_x^R denotes the relevant run threshold, i.e., $\theta_x^R=\underline{\theta}_x$ when $1-k\leq L$ and $\theta_x^R=\theta_x^*$ when $1-k>L$. The terms in (13) are similar to those in (6) in the baseline model, with two main differences. First, as indicated in the first two terms, the bank obtains now a per-unit return $R\left(\theta+x_1\right)$ at date 2 instead of $R\theta$ whenever the loan is not fully repaid. Second, as shown in the third term, the bank is able to obtain the full repayment R in the larger range of values of $\theta\in[1-x_1,2]$ rather than for $\theta\in[1,2]$.

In the presence of a first-loss guarantee scheme with full bankruptcy costs, each bank chooses the underwriting effort $\underline{q}_x$ as a solution to

$$\frac{1}{2}\int_{0}^{\underline{\theta}_x} R\left(\theta+x_1\right)\left(1-\frac{(1-k)}{L}\right) d\theta+\frac{1}{2}\int_{\underline{\theta}_x}^{1-x_1}\left[R\left(\theta+x_1\right)-(1-k)\, r_2\right] d\theta+\frac{1}{2}\int_{1-x_1}^{2}\left[R-(1-k)\, r_2\right] d\theta-cq=0 \quad (14)$$

when $1-k\leq L$ and q_x^* when $1-k>L$ as a solution to

$$\frac{1}{2}\int_{\theta_x^*}^{1-x_1}\left[R\left(\theta+x_1\right)-(1-k)\, r_2\right] d\theta+\frac{1}{2}\int_{1-x_1}^{2}\left[R-(1-k)\, r_2\right] d\theta-\frac{1}{2}\frac{\partial\theta_x^*}{\partial q} q\left[R\left(\theta_x^*+x_1\right)-(1-k)\, r_2\right]-cq=0. \quad (15)$$

The interpretation of the various terms in (14) and (15) is similar to that of (9) and (10). The bank trades-off the marginal benefits of an increase in q, as captured by the first three terms in either expression, with the marginal cost, as measured by cq. As before, the bank remains active until date 2 even in the presence of a run at date 1 only when $1-k\leq L$, as captured by the first term in (14), while, when $1-k>L$, there is an additional effect, reflecting how an increase in q affects the run threshold θ_x^*, as captured by the third term in (15), where $\frac{\partial\theta_x^*}{\partial q}<0$.

In the following proposition, we characterize the effect that the guarantees have on banks' underwriting effort decisions. We use q_x^R to denote either $\underline{q}_x$ or q_x^*, depending on the level of bank capital.

Proposition 4 *For any given level of k, bank underwriting effort increases with the guaranteed amount x_1, while it is independent of x_2: $\frac{dq_x^R}{dx_1}>0$ and $\frac{dq_x^R}{dx_2}=0$. Hence, the introduction of a loan guarantee with $x_1=x_2=x$ increases bank underwriting effort for any level of k: $\frac{dq_x^R}{dx}>0$.*

This proposition highlights that, contrary to what might be expected, the introduction of the loan guarantee induces banks to reduce the riskiness of their portfolios through improved underwriting incentives. The mechanism resembles a "charter value" in that the bank has more to lose when it fails. In fact, the loan guarantee increases the bank's expected profits both through an increase in the probability of survival until date 2 (i.e., a reduction of the threshold θ_x^R) and an increased per-unit return in case of survival. Given this, the bank has stronger incentives to remain active until the final date, which can be achieved through a higher underwriting effort.

It is worthwhile noting that the unambiguously positive effect of the loan guarantee on bank underwriting effort obtains for relatively small values x of the guarantee, so that, as per the assumption that $x < \frac{1-k}{R}$, the guarantee by itself is insufficient to fully cover the promised repayment to depositors. Once the guarantee gets sufficiently large (i.e., if $x \geq \frac{1-k}{R}$), depositors no longer impose any discipline on the bank through the threat of a run. Moreover, the bank may receive a portion of the loan guarantee even when its project fails, with probability $1-q$. In that case, the guarantee would reduce banks' underwriting incentives since the bank's payoff also winds up being partly guaranteed.[16]

Having computed the effect on bank risk-taking and the run threshold, we can now see how the presence of the loan guarantee affects financial stability. While stability has different connotations, ranging from whether viable projects are inefficiently liquidated to the possibility that projects that should be terminated are allowed to continue (an issue we discuss in Section 6), here we focus on the probability the bank is able to carry its projects to maturity, and have those projects deliver positive returns. Specifically, we define a measure of financial stability as the probability that the bank does not fail, and we denote this by γ_x^R as follows:

$$\gamma_x^R \equiv q_x^R \Pr\left(\theta > \theta_x^R\right). \tag{16}$$

The measure γ_x^R combines the notion traditionally studied in the literature on bank runs that focuses primarily on whether the panic run threshold, θ_x^*, increases or falls, with the perspective that banks' actions, through their underwriting effort q, ultimately affect overall stability. We have the following result.

[16] One way of viewing this is that, for x such that $\frac{1-k}{R} > x$, the guarantee effectively represents a state-contingent payment to the bank since it only receives a portion of the guarantee when its project succeeds. Since the success of the bank's projects depends on its underwriting effort, the loan guarantee induces more effort by the bank.

Corollary 1 *Under full bankruptcy costs, the first-loss guarantee* $\mathbf{x} = (x_1, x_2)$ *increases financial stability, i.e.,* $\frac{\partial \gamma_x^R}{\partial x_1} > 0$*.*

The loan guarantee affects financial stability through two channels. First, it has a direct effect on depositors' incentives to run: a larger guarantee x_1 reduces depositors' incentives to run and so the run thresholds θ_x^R decreases. Second, by inducing banks to behave more prudently, the loan guarantee increases the probability that a bank's project is successful thus, ceteris paribus, increasing financial stability. Finally, the increase in q_x^R also further reduces depositors' incentives to run, thus increasing even further the probability that a bank does not fail.

4.2 Bankrupcty-protected first-loss guarantee scheme

In this section, we consider the possibility that the government's transfer to banks, denoted as $\mathbf{x} = (x_1, x_2)$, is sheltered from other frictions which lead to losses resulting from bankruptcy. Specifically, we assume that, in case of default by the bank, depositors receive these amounts even if any revenues stemming from the bank's loans are lost in bankruptcy. This would be consistent with a setting where bankruptcy costs primarily stem from illiquidity associated with selling assets, be they loans or otherwise. The guarantee paid by the government would likely be in cash or other such liquid assets, and less subject to dissipation. As in the previous section, we assume that

$$\frac{1-k}{R} > x_1 \geq x_2, \tag{17}$$

so that the guaranteed amounts are not strictly larger than the (maximum) repayment of 1 that depositors could possibly obtain by withdrawing early, and, in turn, runs are not completely ruled out. The following proposition characterizes depositors' withdrawal decision.

Proposition 5 *The run risk in the presence of a first-loss guarantee* $\mathbf{x} = (x_1, x_2)$ *whose transfers are protected in bankruptcy is as follows:*

a) When $1 - k \leq L$*, runs occur for* $\theta < \underline{\theta}_x^P$*, with* $\underline{\theta}_x^P = \underline{\theta}_x < \underline{\theta}$ *as given by*

$$\underline{\theta}_x^P = \underline{\theta} - x_1.$$

b) When $1 - k > L$*, runs occur for* $\theta < \theta_x^{*P}$*, with* $\theta_x^{*P} < \theta_x^* < \theta^*$ *as given by solution to*

$$\pi_1 = \int_0^{\widehat{n}_x(\theta)} q r_2 dn + \int_{\widehat{n}_x(\theta)}^{\overline{n}} q \frac{Rx_1 \left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn + \int_0^{\overline{n}} (1-q) \frac{Rx_2 \left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn, \tag{18}$$

where $\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n} dn$, *as in Proposition 1. The run threshold* θ_x^{*P} *decreases with* q, k, x_1, *and* x_2: $\frac{\partial \theta_x^{*P}}{\partial q} < 0$, $\frac{\partial \theta_x^{*P}}{\partial k} < 0$, $\frac{\partial \theta_x^{*P}}{\partial x_1} < 0$ *and* $\frac{\partial \theta_x^{*P}}{\partial x_2} < 0$.

As in the case with full bankruptcy costs, the introduction of a first-loss guarantee induces a reduction of the run probability. When the bank is well capitalized, i.e., when $1 - k \leq L$, the fact that the loan guarantee is not lost during the bankruptcy process has no effect on depositors' incentives to withdraw and thus the run threshold is the same as before. This is because the payment accruing to depositors when the bank is insolvent at date 2, either in the case when monitoring pays off (i.e., with probability q) and when it does not (i.e., with probability $1 - q$), is always lower than what they obtain when withdrawing.

By contrast, for poorly capitalized banks, i.e., when $1 - k > L$, the loan guarantee is now more effective in reducing depositors' incentives to run than in the case of full bankruptcy costs, so that $\theta_x^{*P} < \theta_x^*$. The reason is that, in the presence of strategic complementaries in withdrawal decisions, depositors compare the expected payoff at date 1 in the case of runs with that from waiting until date 2, and all payoffs depend now on the others depositors' actions, as captured by the number of withdrawing depositors n – this can be seen from (18). This implies that depositors now take into account the possibility that, depending on the size of n, they may obtain a pro-rata share both at date 1 or date 2, and that the guarantees x_1 and x_2 increase the payoffs they obtain at date 2, as evident in the last two terms on the LHS in (18). This reinforces their incentives to wait until date 2.

As before, the run threshold θ_x^{*P} is decreasing in both q and k. Concerning the guaranteed amounts, both the transfers x_1 and x_2 are preserved in bankruptcy and thus increase the payoffs that depositors receive whenever the bank is unable to make the promised repayment r_2, either when monitoring is effective, with probability q, and they obtain $\frac{Rx_1\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}$, or when it is not, with probability $1 - q$, and they obtain $\frac{Rx_2\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}$. It follows that, in contrast to the case of full bankruptcy costs, the threshold θ_x^{*P} depends now on both transfers x_1 and x_2 in addition to the level of bank capital k. Also, differently from before, the presence of the bankruptcy-protected guarantee changes the sensitivity of the run threshold to the transfer x_1. In particular, while the run threshold θ_x^* in the case of full bankruptcy costs decreases linearly with x_1, the threshold θ_x^{*P} is now more sensitive and decreases even more as x_1 changes, i.e., $\frac{\partial \theta_x^{*P}}{\partial x_1} < \frac{\partial \theta_x^*}{\partial x_1} = -1$. The intuition behind the greater sensitivity lies again on the extra effect of the guarantee in terms of higher payoffs at date 2 when the bank is unable to repay the promised amount because the guaranteed transfers are received by depositors even in bankruptcy. As a result, the sensitivity of θ_x^{*P} to changes in $x = (x_1, x_2)$ and q are not independent, differently from the case when the government transfer is lost in bankruptcy.

The bank's maximization problem for the choice of q is similar to the one characterized in (13), with the only difference that the relevant run threshold is θ_x^P instead of θ_x^R, where $\theta_x^P = \underline{\theta}_x^P$ for $1-k \leq L$ and $\theta_x^P = \theta_x^{*P}$ for $1-k > L$. In the presence of a first-loss bankruptcy protected guarantee scheme, the bank chooses the underwriting effort $\underline{q}_x^P = \underline{q}_x$ as the solution to (14) when $1-k \leq L$, and q_x^{*P} as a solution to

$$\frac{1}{2}\int_{\theta_x^{*P}}^{1-x_1} \left[R\left(\theta + x_1\right) - \left(1-k\right)r_2\right] d\theta + \frac{1}{2}\int_{1-x_1}^{2} \left[R - \left(1-k\right)r_2\right] d\theta - \frac{1}{2}\frac{\partial \theta_x^{*P}}{\partial q} q \left[R\left(\theta_x^{*P} + x_1\right) - \left(1-k\right)r_2\right] - cq = 0 \tag{19}$$

when $1-k > L$. As before, the bank chooses the underwriting effort by trading-off the marginal benefits of an increase in q with the marginal cost. For banks with $1-k \leq L$ the equilibrium value is the same as with the case with full bankruptcy costs as the run threshold is also the same, while it is given by the same expression but with a different run threshold, i.e., θ_x^{*P} instead of θ_x^*, as in (19), when $1-k > L$.

As before, we compute the effect of the introduction of the loan guarantee (x_1, x_2) on the bank's monitoring incentives q. Recall our assumption that there exists a value of capital $\widehat{k}$ such that $\theta^* \to 1$ when $k = \widehat{k}$. We then have the following.

Proposition 6 *Each bank's underwriting effort is independent of the guaranteed amount x_2 ($\frac{dq_x^R}{dx_2} = 0$) when $1-k \leq L$ and decreases with it ($\frac{dq_x^R}{dx_2} < 0$) otherwise. Hence, if $x_1 = x_2 = x$, the following holds:*

a) When $1-k \leq L$, the introduction of the loan guarantee always leads to more bank effort, i.e., $\frac{d\underline{q}_x^P}{dx} > 0$;

b) When $1-k > L$, there exists a value of k denoted as $\widehat{k}_x^P$, with $\widehat{k} < \widehat{k}_x^P < 1-L$, such that introducing the loan guarantee reduces bank effort for $k < \widehat{k}_x^P$, while increasing it as $k \to 1-L$: $\frac{dq_x^{*P}}{dx} < 0$ for $k < \widehat{k}_x^P$ and $\frac{dq_x^{*P}}{dx} > 0$ for $k \to 1-L$.

The proposition shows that the introduction of the loan guarantee has a different effect for low versus high capital banks when the transfer x is not lost in bankruptcy. When banks are very well capitalized (i.e., for $1-k \leq L$), so that they are only exposed to fundamental-driven runs, introducing the loan guarantee induces banks to take less risk and hence choose a higher q. This also occurs for banks that are exposed to panic runs but have a sufficiently high level of capital, i.e., k approaching $1 - L$.

By contrast, when the level of bank capital is sufficiently low, the opposite is true: the introduction of the loan guarantee leads to less effort and more risk-taking. The reason is that, unlike the case studied in Proposition 4, the effect of the government transfer x_2 in the event of project failure is to decrease the bank's incentives for effort.[17] For sufficiently poorly capitalized banks, these effects dominate since these banks are subject to a high run probability and low profits, making the government transfer in failure states relatively important for depositors. As a consequence, poorly capitalized banks reduce their underwriting effort in the presence of guarantees. This result is derived under the assumption that the bank faces extreme run risk for very low values of k (i.e., $\theta^* \to 1$ as $k \to 0$). This ensures that the marginal benefit of the loan guarantee in terms of higher profits when no run occurs approaches zero. When the consequences of a bank run are less severe, as occurs in banks with more capital, or if strategic complementarities are less important, there may not be a region where the introduction of a loan guarantee has a negative incentive effect on bank monitoring.

As in the previous section, it is worth noting that the results in Proposition 6 hold even though, for x small, the guarantee is essentially a state-contingent transfer from the perspective of the bank. In other words, even though the guarantee is not dissipated under bankruptcy, it is still small enough that in the event of failure (i.e., with probability $1 - q$), there is nothing left for the bank. Hence, there is no direct benefit for the bank arising from the loan guarantee in default states. Nevertheless, the introduction of the guarantee can have a negative incentive effect purely through its effect on depositor behavior and, consequently, financial fragility, which here is reflected in the reduced sensitivity of the run threshold θ_x^{*P} to the bank's choice of q.

Finally, from the perspective of the financial stability measure introduced above, $\gamma_x^R \equiv q_x^R \Pr\left(\theta > \theta_x^R\right)$, where q_x^R and θ_x^R refer, respectively, to the relevant level of bank monitoring and depositors' run threshold in equilibrium when the transfer is bankruptcy protected, one can see that for sufficiently well-capitalized banks γ_x^R will be higher as a result of the introduction of the loan guarantee. For

[17] This occurs because the component x_2 of the loan guarantee reduces the sensitivity of the run threshold to changes in q and, at the same time, by reducing the run threshold, it reduces the losses associated with an increase in the probability of a run due to low monitoring incentives by the bank. These two effects combined lead the bank to have lower incentives to exert effort q the higher is x_2.

poorly capitalized banks, namely those for which $k < \widehat{k}_x^P$ as defined in Proposition 6, the effect of the introduction of a loan guarantee is ambiguous on financial stability since there are two countervailing forces at play. On the positive side, ceteris paribus the guarantee scheme reduces the likelihood of a run by depositors. However, on the negative side, bank monitoring decreases for banks with very low capital. The overall effect is therefore the combination of these two forces, and depends on which source of risk – poor loan performance or run risk – is more important for the bank.

4.3 Guarantee scheme with loss-sharing

Our results above on the incentive effects of a loan guarantee scheme hold for the case where the government is in a first-loss position, and insures losses to banks up to an amount Rx. In this section, we show that those qualitative results extend to the other main type of loan guarantee scheme being employed, namely one of loss sharing between the government and the bank. Specifically, suppose that the government commits to cover a fraction $y \in (0,1)$ of bank losses $R(1-\theta)$ and, as a result, the bank per unit loan return is equal to $max\left\{R, R\theta + R\left(1-\theta\right)y\right\}$.

Similarly to the first-loss guarantee scheme illustrated in the previous section, by increasing banks' loan return, the support offered by the government affects both depositors' incentives to run and the banks' risk-taking decision. We have the following result, which summarizes both the case with full bankruptcy costs and the one where transfer are bankruptcy-protected for the case when $y_1 = y_2$.

Proposition 7 *The introduction of a loss-sharing guarantee with $y_1 = y_2$ leads to the following:*

a) In the case of full bankruptcy costs:

1. *Runs occur for $\theta < \underline{\theta}_y$ when $1-k \leq L$, and for $\theta < \theta_y^*$ when $1-k > L$, where $\theta_y^* > \underline{\theta}_y$ are given, respectively, by*

$$\underline{\theta}_y = \frac{\underline{\theta} - y}{1-y} \text{ and } \theta_y^* = \frac{\theta^* - y}{1-y}. \tag{20}$$

2. *For any level of k, each bank's underwriting effort q_y^R increases in the guaranteed amount: $\frac{dq_y^R}{dy} > 0$.*

b) In the case where the government's transfers are protected from bankruptcy:

1. *Runs occur for* $\theta < \underline{\theta}_y^P = \underline{\theta}_y$ *when* $1-k \leq L$ *and for* $\theta < \theta_y^{*P}$ *when* $1-k > L$*, where* $\theta_y^{*P} > \underline{\theta}_y^P$ *solves*

$$\pi_1 = \int_0^{\widehat{n}_y(\theta)} qr_2 dn + \int_{\widehat{n}_y(\theta)}^{\overline{n}} q \frac{R(1-\theta)\,y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn + \int_0^{\overline{n}} (1-q)\frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn,$$

2. *Each bank's effort* q_y^R *increases with the introduction of the guarantees when* $1-k \leq L$*:* $\frac{dq_y}{dy} > 0$*. Moreover, there exists a value* $\widehat{k}_y^P \in \left(\widehat{k}_y, 1-L\right)$ *such that* q_y^R *decreases after the introduction of guarantees for any* $k < \widehat{k}_y^P$*:* $\frac{dq_y^*}{dy} < 0$*.*

As stated in the proposition, the scheme where the guarantee requires banks to share any losses on a proportional basis delivers the same qualitative results in terms of financial fragility and bank underwriting efforts as the scheme where the guarantee represents a first-loss position for the government. As before, irrespective of whether the guaranteed amounts y_1 and y_2 are lost or protected in bankruptcy, banks are more likely to experience runs when they are poorly capitalized (i.e., when $1-k > L$) than when they are well capitalized (i.e., when $1-k \leq L$), as they also experience panic runs in additional to fundamental-driven ones. Also, for any level of bank capitalization, the guarantee reduces the run threshold relative to the case with no guarantees. As before, the effect of the loan guarantee on bank monitoring incentives depends on the treatment of the guarantee in bankruptcy and the level of capitalization. With full bankruptcy costs, the bank increases its underwriting effort when a guarantee is introduced, for any capital level. By contrast, the bank responds to the introduction of a bankruptcy-protected guarantee by decreasing its underwriting effort when it has little capital, i.e., when $k < \widehat{k}_y^P$, in the range $1-k > L$ (this again makes use of the assumption that there exists a value of capital $\widehat{k}$ such that $\theta^* \to 1$ when $k = \widehat{k}$). Note as well that the implications for financial stability, as measured by γ (see (16)), are qualitatively the same as in Section 4.2.

4.4 Comparison of loan guarantee schemes

As shown above, the two guarantee schemes - first-loss or loss-sharing - deliver qualitatively similar results, although their designs are quite different. While this suggests that our main findings are robust to how exactly the guarantee may be designed, a natural question that arises is whether one particular style of guarantee may be more effective in terms of its effects on the bank's underwriting effort and costs it entails. In other words, can the government do better by implementing one type of loan guarantee scheme rather than another?

In this section, we compare the two guarantee schemes (GS) analyzed in the previous sections, under the maintained assumption that the guaranteed amount is lost in bankruptcy. For tractability, we focus on the case where the bank is well capitalized, i.e., $1-k \leq L$, so that panic-based runs do not occur and, as shown above, the introduction of the loan guarantee unambiguously improves financial stability. Our primary aim is to assess which loan guarantee scheme is more cost-effective and which, if any, provides a stronger incentive effect for the bank. In what follows, we use the subscript x when referring to the guarantee scheme in which the government is in a first-loss position (GS_x) and the subscript y to denote the one entailing loss sharing between the government and the bank (GS_y).

When a bank has a level of capital satisfying $1-k \leq L$, the relevant run thresholds $\underline{\theta}_x$ and $\underline{\theta}_y$ in the guarantee scheme GS_x and GS_y are given, respectively, by $\underline{\theta}_x = \underline{\theta} - x$ (see (12)) and $\underline{\theta}_y = \frac{\underline{\theta}-y}{1-y}$ (see (20)). To compare the two schemes, we consider the case where the sizes x and y of the guarantees are set, all things equal, to lead to the same run threshold: $\underline{\theta}_x = \underline{\theta}_y$. Equating these two, we specify $\underline{y}$ as the level of y for which the two guarantee schemes implement the same probability of a run. Hence, $\underline{y}$ solves

$$\frac{\underline{\theta}-y}{1-y} = \underline{\theta} - x,$$

and is equal to

$$\underline{y} = \frac{x}{1-\max\{\underline{\theta}-x, 0\}} \geq x,$$

since $\underline{\theta} - x \equiv \underline{\theta}_x < 1$.

For a given size of the guarantee x, the guarantee scheme GS_x entails a disbursement for the government equal to

$$\begin{aligned} GD_x &= \int_0^{\underline{\theta}-x} Rx\left(1-\frac{1-k}{L}\right)d\theta + \int_{\underline{\theta}-x}^{1-x} Rxd\theta + \int_{1-x}^{1} R\left(1-\theta\right)d\theta \\ &= Rx - \frac{Rx^2}{2} - Rx\left(\underline{\theta}-x\right)\frac{1-k}{L}, \end{aligned} \tag{21}$$

while GS_y entails a disbursement equal to

$$\begin{aligned} GD_y &= \int_0^{\underline{\theta}-x} R\left(1-\theta\right)y\left(1-\frac{1-k}{L}\right)d\theta + \int_{\underline{\theta}-x}^{1} R\left(1-\theta\right)y \\ &= \frac{Ry}{2} - Ry\left(\underline{\theta}-x\right)\frac{1-k}{L} + \frac{Ry}{2}\frac{1-k}{L}\left(\underline{\theta}-x\right)^2 \end{aligned} \tag{22}$$

Comparing GD_x and GD_y when $y = \underline{y}$, we have the following result.

Proposition 8 *For any $x > 0$, the guarantee scheme in which the government is in a first-loss position entails a larger disbursement for the government than when the bank and the government share losses, when both schemes are designed to achieve the same run threshold, but induces the bank to choose a higher q.*

The proposition shows that while the first-loss guarantee scheme, GS_x, provides greater incentives to the bank through improved bank underwriting standards, it achieves this at a higher cost. Hence, neither type of scheme unambiguously dominates the other: the cost-minimizing scheme, GS_y, is also not as effective at improving banking sector stability. As such, our findings here once again reinforce the view that guarantee design features, while perhaps important, are unlikely to dramatically affect the main qualitative conclusions concerning the stability effects of loan guarantees.

5 Deposit insurance

Our analysis has focused on guarantees that are directly tied to bank lending, and that insure the bank's loan portfolio, or individual loans, against default risk by borrowers. These guarantees are primarily viewed as stimulative instruments, having been used in response to crises (e.g., Covid-19) that worsen economic fundamentals and may make banks unwilling to lend, thus possibly worsening the crisis. However, government guarantees are common in other contexts and, for banks in particular, are evident in deposit insurance schemes in most countries. Such guarantees also contain a stimulative component since, in addition to reducing the required interest rate that must be paid to depositors, they also directly increase stability by reducing depositors' run risk (see, e.g., Allen et al., 2018). In this section, we contrast the effect of a deposit guarantee on banks' risk-taking incentives and financial stability to that of the loan guarantees studied above, and show that, while sharing some similarities, the ultimate effects of the two types of guarantees are quite different.

Specifically, consider a deposit guarantee scheme that ensures depositors receive a minimum repayment $\delta > 0$ if the bank is insolvent and cannot fully repay depositors' claims. To keep things simple, we assume that $0 < \delta < L < 1$ so that the guarantee only needs to be paid when depositors do not run, making it comparable to the analysis with loan guarantees.[18] As above, we start by

[18]The assumption that the guarantee is only paid at date 2 when depositors do not run is without loss of generality. As shown in Allen et al. (2018), the run threshold decreases in the guaranteed amount δ even when this is paid in

solving depositors' withdrawal decisions. We have the following result.

Proposition 9 *The run risk in the presence of deposit insurance depends on the level of bank capitalization.*

a) When $1-k \leq L$, runs are only triggered by bank insolvency at date 2 and they occur when θ falls below the threshold $\underline{\theta}_\delta(k) = \underline{\theta}(k)$, as given in (4).

b) When $1-k > L$, runs are also driven by panics and they occur when θ falls below the threshold $\theta^*_\delta(q,k,\delta) < \theta^*(q,k)$, equal to

$$\theta^*_\delta(q,k,\delta) = \frac{(1-k)\,r_2}{R}\frac{(qr_2 - \pi_1) + \delta\,(1-q)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)}, \tag{23}$$

where $\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)n} dn$. The threshold $\theta^*_\delta \in (\underline{\theta}, 1)$ decreases with q, k, and δ: $\frac{\partial \theta^*_\delta(q,k)}{\partial q} < 0$, $\frac{\partial \theta^*_\delta(q,k)}{\partial k} < 0$, and $\frac{\partial \theta^*_\delta(q,k)}{\partial \delta} < 0$.

In the case of highly capitalized banks, i.e., those with $1-k \leq L$, the run threshold remains the same as with no guarantees. The reason is that in this case the threshold is derived from the bank's insolvency condition as given in (2), which is unaffected by the presence of deposit insurance. By contrast, for banks with capital such that $1-k > L$, the run threshold is now reduced relative to the case with no guarantees because the transfer δ increases what depositors expect to receive at date 2, thus reducing their incentives to withdraw prematurely.

Having characterized the effect of the deposit guarantee on the run thresholds, we now move on to see how it affects banks' risk-taking decisions. Differently from the loan guarantees, banks do not directly benefit from the introduction of the deposit insurance as the public funds are transferred to depositors. However, banks may benefit indirectly from a deposit guarantee as long as it reduces their exposure to runs, thus also affecting their underwriting incentives, q.

Similarly to above, denote as $\theta^R_\delta = \{\underline{\theta}_\delta, \theta^*_\delta\}$ the relevant run threshold, which depends on whether $1-k \gtrless L$. Given θ^R_δ, we can now solve for a bank's underwriting incentives. Each bank chooses q to maximize

$$\frac{1}{2}q\int_0^{\theta^R_\delta} \max\left\{0, R\theta\left(1-\frac{1-k}{L}\right)\right\} d\theta + \frac{1}{2}q\int_{\theta^R_\delta}^1 \left[R\theta - (1-k)\,r_2\right] d\theta + \frac{1}{2}q\int_1^2 \left[R - (1-k)\,r_2\right] d\theta - \frac{cq^2}{2}.$$

The interpretation of the terms in the expression for bank profits is as in the case without guarantees. A bank earns positive profits if there is no run and the project succeeds with probability q. A

the event of a run.

highly capitalized bank also accrues positive profits in the event of a run as it does not have to liquidate its entire portfolio to meet depositors' withdrawals. As a result, it receives the return $R\theta$ with probability q on the $\left(1-\frac{1-k}{L}\right)$ share of its portfolio that is not liquidated. Importantly, and differently from the case of loan guarantees, the presence of deposit insurance does not directly increase the payoff that the bank obtains at date 2. The bank's optimal choice of q then solves

$$\frac{1}{2}\int_0^{\theta_\delta^*} R\theta\left(1-\frac{1-k}{L}\right)d\theta+\frac{1}{2}\int_{\theta_\delta^*}^1\left[R\theta-(1-k)\,r_2\right]d\theta+\frac{1}{2}\int_1^2\left[R-(1-k)\,r_2\right]d\theta-cq=0,$$

when $1-k\leq L$ and

$$\frac{1}{2}\int_{\theta_\delta^*}^1\left[R\theta-(1-k)\,r_2\right]d\theta+\frac{1}{2}\int_1^2\left[R-(1-k)\,r_2\right]d\theta-\frac{1}{2}\frac{\partial\theta_\delta^*}{\partial q}q\left[R\theta_\delta^*-(1-k)\,r_2\right]-cq=0, \quad (24)$$

when $1-k>L$. We have the following result.

Proposition 10 *The introduction of a deposit guarantee scheme induces low capitalized banks to reduce their underwriting effort, while it has no effect on highly capitalized banks, i.e.,* $\frac{dq_\delta^*}{d\delta}<0$ *when* $1-k>L$ *and* $\frac{dq_\delta}{d\delta}=0$ *when* $1-k\leq L$.

The proposition shows that banks' underwriting incentives are affected differently by the deposit guarantee depending on the level of bank capitalization. Highly capitalized banks, i.e., those for which $1-k\leq L$, are not affected by the introduction of the deposit guarantee since the run threshold $\underline{\theta}_\delta$ does not depend on δ. Hence, it follows immediately that when $1-k\leq L$, the presence of a deposit guarantee has no effect on the choice of q. By contrast, less capitalized banks that are subject to panic runs respond to the introduction of the guarantee by taking more risk, i.e., choosing a lower q. The intuition is that the presence of deposit insurance reduces depositors' incentives to run. It also reduces the sensitivity of the run threshold to changes in underwriting effort. Overall, these effects reduce the benefit the bank obtains when choosing a high underwriting effort. As a result of the lower underwriting effort, the introduction of the deposit guarantee may have an ambiguous impact on financial stability. On the one hand, by reducing depositors' incentives to run, it increases financial stability. On the other hand, by creating incentives for banks to reduce their underwriting standards, it weakens the beneficial effect on depositors' withdrawal decision and, at the same time, increases the bank's failure probability.

Overall, the result in Proposition 10 highlights the difference between guarantees introduced as deposit insurance schemes and those in the form of loan guarantees. In line with the idea that insurance mechanisms induce moral hazard considerations, the presence of deposit insurance never improves banks' underwriting incentives, and it rather reduces them for poorly capitalized banks, i.e., those for whom $1-k \leq L$. This stands in contrast with the results obtained in the case of loan guarantees. Irrespective of the precise type of the scheme, first-loss or loss-sharing, the introduction of loan guarantees tends to improve banks' monitoring incentives, with the exception of the case where guarantees are bankruptcy-protected and banks have very little capital (i.e., $k \in [0, \widehat{k}_y^P)$).

6 Inefficient liquidation and zombie lending

So far we have focused on the effects of loan guarantees on banks' incentives to monitor borrowers. In this section, we analyze banks' incentives to engage in "evergreening", or in other words inefficient loan continuation, and study how these interact with run risk and are affected by loan guarantees. To do so, we modify the model slightly and assume that at date 1 a bank can choose whether to liquidate its loan portfolio or continue until the final date. Such choice is made after depositors' withdrawal decision and thus does not interfere with how depositors evaluate their private signals. However, it may affect depositors' run decisions as they correctly anticipate the bank's optimal liquidation decision.

We start by analyzing banks' liquidation decision at date 1 in a setting where withdrawing at $t = 1$ is not possible so that there are no runs. In this case, each bank compares the loan's expected return at date 2 with its liquidation value at date 1, net of depositors' repayments, and chooses to liquidate if θ falls below the threshold θ_L^B as given by the solution to

$$L - (1-k)r_2 = q\left(R\theta - (1-k)r_2\right),$$

which is equivalent to

$$\theta_L^B = \frac{L - (1-q)(1-k)r_2}{qR}. \tag{25}$$

The question is whether banks' liquidation decision is optimal from a social perspective. A social planner finds it optimal to liquidate the portfolio when θ falls below the threshold θ_L^{SP}, which is given by the solution to

$$L = q\theta R,$$

and is thus equal to

$$\theta_L^{SP} = \frac{L}{qR}. \tag{26}$$

We have the following result.

Lemma 1 *If withdrawing at $t = 1$ is not possible, we have that $\theta_L^B < \theta_L^{SP}$. In addition, the difference $\theta_L^{SP} - \theta_L^B$ decreases in k: $\frac{\partial(\theta_L^{SP} - \theta_L^B)}{\partial k} < 0$.*

The lemma shows that for $\theta \in (\theta_L^B, \theta_L^{SP}]$ banks have an incentive to engage in evergreening, that is, to continue projects that instead should be liquidated at date 1. Consistent with empirical findings (see e.g., Blattner et al., 2021; Schivardi et al., 2021), the extent to which banks evergreen loans decreases with the level of bank capital.

We now go back to the case where depositors can withdraw at $t = 1$. This implies that loans can be liquidated at date 1 for two reasons: either because a run occurs, or because a bank prefers to liquidate its portfolio prematurely even if no run occurs. To see when either case is relevant, we compare banks' liquidation threshold θ_L^B with the run threshold $\theta^R = \{\underline{\theta}, \theta^*\}$.

Lemma 2 *The comparison between θ_L^B and θ^R depends on the level of bank capital k. In particular, $\theta_L^B \leq \theta^R$ for $k \leq \overline{k}_L$ and $\theta_L^B > \theta^R$ for $k > \overline{k}_L$, where $\overline{k}_L = 1 - \frac{L}{r_2} > 1 - L$.*

The lemma illustrates the relevance of banks' liquidation policy in the model. When a bank has little capital and is, thus, exposed to panic runs, it never finds it optimal to liquidate the portfolio at date 1 when a run does not occur. In other words, given $\theta^* > \theta_L^B$, banks' liquidation decision is not relevant for poorly capitalized banks as the fragility stemming from depositors' run decisions leads already to more liquidation than what a bank would prefer.

The case for better capitalized banks, who are only exposed to fundamental runs, is different. In this case, banks with $k > \overline{k}_L$, and thus $\theta_L^B > \underline{\theta}$, liquidate their portfolios for $\theta \in [0, \theta_L^B]$. By contrast, those with $k < \overline{k}_L$ experience a run for any $\theta < \theta_L^B < \underline{\theta}$. It follows that the entire investment is liquidated for $\theta < [0, \theta_L^B]$, while only partial liquidation takes place as a consequence of a run for $\theta < [\theta_L^B, \underline{\theta}]$.

Now that we have seen how depositors' run decisions interact with banks' liquidation decision, we can analyze the extent to which evergreening occurs. To this end, we compare the thresholds θ_L^B and $\theta^R = \{\underline{\theta}, \theta^*\}$ of banks' liquidation decision and depositors' run behavior, respectively, with the liquidation threshold of the planner as given by θ_L^{SP} in (26). We have the following result.

Lemma 3 *In an economy with runs, $\theta_L^{SP} \geq \max\left\{\theta_L^B, \underline{\theta}\right\}$ for $1 - k \leq L$ and $\theta^* > \theta_L^{SP} > \theta_L^B$ for $1 - k > L$.*

The lemma shows that liquidation in the baseline economy is always inefficient. Highly capitalized banks with $1-k \leq L$ don't liquidate enough, thus carrying over until the final date projects that would be optimal to liquidate at $t=1$. On the contrary, for low capital banks with $1-k > L$, there is excessive liquidation. For these banks, however, the excessive liquidation stems from depositors' behavior (i.e., panic runs) rather than from excessively stringent decisions by the banks. The result is illustrated in Figure 5.

Insert Figure 5

6.1 Introducing the loan guarantee

In this section, we analyze the effect of the introduction of a loan guarantee scheme on the incidence of evergreening. We follow the same steps as in the previous section. Hence, we first identify the threshold θ_{Lx}^{B} below which liquidating early is optimal for banks and we then compare it with the run thresholds $\theta_x^R = \{\underline{\theta}_x, \theta_x^*\}$ as given in (12). The cutoff θ_{Lx}^{B} is equal to the solution to

$$L-(1-k)r_2 = q\left(R(\theta+x)-(1-k)r_2\right),$$

and thus

$$\theta_{Lx}^{B} = \frac{L-(1-q)(1-k)r_2}{qR} - x = \theta_L^B - x.$$

It is easy to see that the comparison between θ_{Lx}^{B} and θ_x^R is as in Lemma 2 so that $\theta_x^R > \theta_{Lx}^{B}$ for banks with $k < \overline{k}_L$.

We can now study the effect of the introduction of the guarantee on banks' evergreening incentives.

Proposition 11 *In an economy with loan guarantees and full bankruptcy costs:*

a) when $1-k \leq L$, the difference $\theta_L^{SP} - \max\left\{\theta_{Lx}^{B}, \underline{\theta}_x\right\}$ is larger than in the case without guarantees;

b) when $1-k > L$, there exists a level of capital $\widetilde{k}_L \in [0, 1-L)$ such that $\theta_x^* \geq \theta_L^{SP}$ for $k \leq \widetilde{k}_L$ and $\theta_x^* < \theta_L^{SP}$ for $k > \widetilde{k}_L$.

The proposition, which is illustrated in Figure 5, shows that the presence of the loan guarantee worsens the evergreening problem. For highly capitalized banks, for which $1-k \leq L$, the presence of the guarantee increases the range of values of the fundamental θ for which there is inefficient loan continuation. More importantly, banks exposed to panic runs, with capital $\widetilde{k}_L < k < 1 - L$, start evergreening as the loan guarantee reduces the panic run threshold to a value below the threshold for liquidation of the social planner. Hence, these banks now have an incentive to evergreen loans, even though they would not have done so in the absence of the loan guarantee. This result is specific to loan guarantees and would not hold following the introduction of deposit insurance, as in Section 5. In that case, deposit insurance does not affect the liquidation decision of the bank and the fundamental run threshold. This implies that banks with high levels of capital would not change their incentives to evergreen. For banks with less capital that are exposed to panic runs, the run threshold would decrease but not sufficiently to fall below the social planner's liquidation threshold.[19]

7 Conclusion

In this paper, we presented a model in which banks raise demandable deposits and grant long-term loans. Banks' expected return depends on the economy fundamentals, as well as on bank underwriting efforts. Our focus has been to analyze how the introduction of loan guarantees affects bank incentives and financial fragility. As with any form of insurance, the introduction of a loan guarantee reduces depositors' run probability. The reason is that the guarantee increases the range in which the bank is able to make the promised repayment to depositors in the final date and, if the government transfers are bankruptcy-protected, it also increases depositors' expected payoffs at the final date. Both of these effects reduce depositors' incentives to withdraw prematurely, thus reducing financial fragility.

We also show that, contrary to perceived wisdom, introducing loan guarantees improves banks' monitoring incentives in many instances. This finding arises both from a direct (positive) effect of loan guarantees and an indirect effect from the reduction in run probability. The result may differ only when the loan guarantee is shielded from bankruptcy costs. In this case, the presence of a loan guarantee reduces the sensitivity of the run threshold to changes in the underwriting effort. This last effect is negative as it reduces the benefit for the bank from increasing its effort, and it may dominate when banks are insufficiently capitalized.

In our framework, depositors suffer from potential coordination failures which lead them to run on their bank. However, we abstract from other considerations in the literature on bank runs,

[19] Formal results on evergreening in the presence of deposit insurance are available from the authors upon request.

such as the need to provide liquidity insurance to depositors if they are risk averse. Nevertheless, we believe our framework, which incorporates credit and run risk, could be extended to study the interaction of loan guarantees with liquidity provision, the focus of much of the literature on financial fragility.

In our setting, banks, or bank owners, are the primary decision-makers concerning bank portfolio choice and rate-setting policies. By making banks the residual claimant, we are able to study the role of capital for banks, and how that relates to both credit risk and financial fragility, an issue that for the most part has been absent in the literature. In doing so, however, we take banks' capital structures as given. An interesting avenue for future research would be to endogenize banks' capital structures and consider how the incentives to raise capital may be driven by the two sources of risk we highlight here, as well as how government guarantees, such as for loans, influence this choice.

8 References

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9 Appendix

Proof of Proposition 1: The proof proceeds in steps. First, we pin down the threshold $\underline{\theta}(k)$, which corresponds to the upper bound of the lower dominance region, as characterized in the main text. Second, we characterize the threshold $\theta^*(q,k)$ summarizing depositors' withdrawal decision in the intermediate range of fundamentals, i.e., when $\theta \in [\underline{\theta}(k), \overline{\theta})$. Third, we show that for any $1-k \leq L$, the relevant run threshold is $\underline{\theta}(k)$, while it is $\theta^*(q,k) > \underline{\theta}(k)$ for any $1-k > L$. We conclude the proof with the comparative statics for the two run thresholds with respect to q, L and k.

The lower dominance region corresponds to the range of fundamentals θ in which running is a dominant strategy and, as such, is independent of other depositors' withdrawal decisions. Given the existence of bankruptcy costs, a depositor's expected repayment at date 2 is $qr_2 > 1$ when the bank is solvent, and 0 otherwise. Thus, the threshold $\underline{\theta}(k)$ corresponds to the level of fundamentals at which the bank is solvent, i.e., where the bank has just enough resources to repay r_2 to all $1-k$ depositors. Formally, this corresponds to the solution to

$$R\theta - (1-k)\, r_2 = 0. \tag{27}$$

Solving (27) for θ, we obtain condition (4) in the proposition.

For $\theta \in (\underline{\theta}(k), \overline{\theta})$, a depositor's withdrawal decision depends on what other depositors do when $1-k > L$. Assume that all depositors behave according to the threshold strategy s^*. Then, the fraction of depositors withdrawing at date 1, $n(\theta, s^*)$, is equal to the probability of receiving a signal below s^* and can be specified as follows:

$$n(\theta, s^*) = \begin{cases} 1 & \text{if } \theta \leq s^* - \varepsilon \\ \frac{s^* - \theta + \varepsilon}{2\varepsilon} & \text{if } s^* - \varepsilon < \theta \leq s^* + \varepsilon \\ 0 & \text{if } \theta > s^* + \varepsilon \end{cases}.$$

Depositors' withdrawal decisions are characterized by the pair $\{s^*, \theta^*\}$, which corresponds to the solution to the following system of equations:

$$R\theta \left(1 - \frac{n(\theta, s^*)(1-k)}{L}\right) - (1 - n(\theta, s^*))(1-k)\, r_2 = 0, \tag{28}$$

and

$$qr_2 \Pr(\theta > \theta^* | s^*) = 1 \Pr(\theta > \theta_n | s^*) + \frac{L}{(1-k)\, n(\theta, s^*)} \Pr(\theta < \theta_n | s^*), \tag{29}$$

where $\theta_n = s^* + \varepsilon - 2\varepsilon \frac{L}{1-k}$ represents the level of θ for which the bank liquidates the entire portfolio at date 1 and, thus, is equal to the solution to

$$n(\theta, s^*)(1-k) = L.$$

Condition (28) identifies the level of fundamentals, θ^*, at which the bank is at the brink of insolvency at date 2 when $n(\theta^*, s^*) > 0$ depositors run, for given s^*. Condition (29) is depositors' indifference condition: the LHS represents a depositor's expected utility from withdrawing at date 2, while the RHS represents the expected utility from withdrawing at date 1. This condition pins down s^* given $\theta^*(s^*)$ from (28), so that together the two equations characterize the equilibrium withdrawal decisions $\{s^*, \theta^*\}$.

Differentiating (28) with respect to θ and n, we obtain, respectively,

$$R\left(1 - \frac{n(\theta, s^*)(1-k)}{L}\right) - \frac{\partial n(\theta, s^*)}{\partial \theta}\left[R\theta\frac{(1-k)}{L} - (1-k)\, r_2\right] > 0,$$

and

$$-R\theta\frac{1-k}{L} + (1-k)\, r_2 < 0,$$

for any $\theta > \underline{\theta}(k)$ since $1-k > L$ and $\frac{\partial n(\theta,s^*)}{\partial\theta} < 0$. Since $n(\theta, s^*)$ is a decreasing function of θ, it follows that the LHS in (28) strictly increases in θ and so does the expected utility at date 2. Furthermore, rearranging (28) as follows

$$R\theta - (1-k)\, r_2 - n(\theta, s^*)\left(R\theta\frac{(1-k)}{L} - (1-k)\, r_2\right) = 0,$$

it is easy to see that the expression in (28) is always negative when $\theta \le \underline{\theta}(k)$ for any $n(\theta, s^*) > 0$ and positive for any $\theta \ge \overline{\theta}$. This also implies that a depositor's expected utility differential between withdrawing at date 2 and date 1, which corresponds to the difference between the LHS and RHS in (29), is also increasing in θ, negative for $\theta < \underline{\theta}(k)$ and positive when $\theta \ge \overline{\theta}$. It follows that a unique threshold s^* exists at which a depositor is indifferent between withdrawing at date 2 or at date 1.

To obtain the expression for $\theta^*(q, k)$ as in the proposition, we perform a change of variable by defining $\theta^*(n) = s^* + \varepsilon(1 - 2n)$. At the limit when $\varepsilon \to 0$, $\theta^*(n) \to s^*$ and we denote the run threshold as $\theta^*(q,k)$, which corresponds to the solution to

$$\int_0^{\widehat{n}(\theta)} qr_2 dn - \int_0^{\overline{n}} dn - \int_{\overline{n}}^1 \frac{L}{(1-k)\,n} dn = 0, \tag{30}$$

where $\widehat{n}(\theta^*)$ solves (3) and $\overline{n}$ solves

$$(1-k)n = L.$$

The expression in (5) is obtained by rearranging the terms, using $\underline{\theta} = \frac{(1-k)r_2}{R}$, and denoting

$$\pi_1 = \int_0^{\overline{n}} dn + \int_{\overline{n}}^1 \frac{L}{(1-k)\,n} dn. \tag{31}$$

Now, we move on to show that the relevant run threshold is $\underline{\theta}(k)$ when $1-k \leq L$ and $\theta^*(q,k)$ when $1-k > L$. Consider first the case in which $1-k \leq L$. When $1-k = L$, $\pi_1 = 1$ and (28) simplifies to

$$(1 - n(\theta^*, s^*))\,[R\theta - (1-k)\,r_2]\,,$$

which is positive for $\theta > \underline{\theta}(k)$ and negative for $\theta < \underline{\theta}(k)$ for any $n(\theta^*, s^*) < 1$. Then, from (29), it follows that running is optimal when $\theta < \underline{\theta}(k)$, irrespective of $n(.)$. Hence, the relevant run threshold is $\underline{\theta}(k)$ when $1-k = L$. Since $\underline{\theta}(k)$ is decreasing in $1-k$, condition (28) becomes less binding for any n when $1-k$ falls below L. This implies that $\underline{\theta}(k)$ is still the relevant run threshold when $1-k < L$.

Consider now the case where $1-k > L$. Since (28) is increasing in θ for any $\theta > \underline{\theta}(k)$, it follows that $\theta^*(q,k) > \underline{\theta}(k)$ when $1-k > L$.

To complete the proof, we compute $\frac{\partial \underline{\theta}(k)}{\partial k}$, $\frac{\partial \theta^*(q,k)}{\partial q}$, $\frac{\partial \theta^*(q,k)}{\partial L}$, and $\frac{\partial \theta^*(q,k)}{\partial k}$. Differentiating (4) with respect to k, we obtain

$$\frac{\partial \underline{\theta}(k)}{\partial k} = -\frac{r_2}{R} < 0.$$

Using (5), we compute the effect of q, L, and k on $\theta^*(q,k)$ as follows:

$$\begin{aligned}\frac{\partial \theta^*(q,k)}{\partial q} &= \frac{\underline{\theta}}{\left(qr_2 - \pi_1 \frac{(1-k)}{L}\right)^2}\left\{r_2\left(qr_2 - \pi_1\frac{(1-k)}{L}\right) - r_2\left(qr_2 - \pi_1\right)\right\} \\ &= -\frac{\underline{\theta} r_2 \pi_1}{\left(qr_2 - \pi_1 \frac{(1-k)}{L}\right)^2}\left[\frac{(1-k)}{L} - 1\right] < 0,\end{aligned}$$

$$\frac{\partial \theta^*(q,k)}{\partial L} = \frac{\underline{\theta}}{\left(qr_2 - \pi_1 \frac{(1-k)}{L}\right)^2}\left\{-\frac{\partial \pi_1}{\partial L}\left(qr_2 - \pi_1\frac{(1-k)}{L}\right) + (qr_2 - \pi_1)\,\frac{(1-k)}{L}\left[\frac{\partial \pi_1}{\partial L} - \frac{\pi_1}{L}\right]\right\} < 0,$$

and

$$\frac{\partial\theta^*(q,k)}{\partial k} = \frac{1}{\left(qr_2 - \pi_1\frac{(1-k)}{L}\right)}\left\{\frac{\partial\underline{\theta}}{\partial k}(qr_2-\pi_1) - \underline{\theta}\frac{\partial\pi_1}{\partial k} + \frac{\theta^*}{L}\left[\frac{\partial\pi_1}{\partial k}(1-k) - \pi_1\right]\right\} < 0,$$

with $\frac{\partial\pi_1}{\partial L} = \int_{\overline{n}}^1 \frac{1}{(1-k)n}dn > 0$, $\frac{\partial\pi_1}{\partial k} = \int_{\overline{n}}^1 \frac{L}{(1-k)^2 n}dn > 0$ and $\frac{\partial\pi_1}{\partial L} - \frac{\pi_1}{L} = -\frac{1}{L}\int_0^{\overline{n}} dn$. Hence, the proposition follows. □

Proof of Proposition 2: Using backward induction, we first compute the optimal q and then solve for r_2. Concerning the choice of q, (9) and (10) are obtained by differentiating (6) with respect to q, setting $\theta^R = \underline{\theta}$ when $1-k \le L$ and $\theta^R = \theta^*$ when $1-k > L$, respectively.

We now move to the choice of r_2. Consider first the case when $1-k \le L$ when the relevant run threshold is $\underline{\theta}$. Since $\frac{\partial\underline{\theta}}{\partial r_2} > 0$ and a higher r_2 reduces bank's profits when no runs occur, it is optimal for the bank to choose the lowest possible r_2, which corresponds to the solution of (7) holding with equality.

Consider now the case when $1-k > L$. In this case, the above argument does not apply since $\frac{\partial\theta^*}{\partial r_2} < 0$ may hold. Therefore, we write the Lagrangian for the bank's problem as

$$\mathcal{L} = \Pi|_{q=q^*} - \mu\left\{1 - \frac{1}{2}\int_0^{\theta^*}\frac{L}{1-k}d\theta - \frac{1}{2}\int_{\theta^*}^2 qr_2 d\theta\right\},$$

where Π is given in (6). The Kuhn-Tucker conditions are

$$\begin{aligned} &-\frac{1}{2}\frac{\partial\theta^*}{\partial r_2}\left[R\theta^* - (1-k)r_2\right] - \frac{1}{2}\int_{\theta^*}^2 q^*(1-k)\,d\theta + \frac{\partial\Pi}{\partial q}\frac{dq^*}{dr_2} + \frac{1}{2}\mu\int_{\theta^*}^2 q^* d\theta \\ &-\frac{1}{2}\mu\left[\frac{\partial\theta^*}{\partial r_2} + \frac{\partial\theta^*}{\partial q}\frac{dq^*}{dr_2}\right]\left[q^* r_2 - \frac{L}{1-k}\right], \end{aligned} \tag{32}$$

$$\mu\left\{1 - \frac{1}{2}\int_0^{\theta^*}\frac{L}{1-k}d\theta - \frac{1}{2}\int_{\theta^*}^2 qr_2 d\theta\right\} = 0,$$

$$\mu \ge 0.$$

The derivative $\frac{dq^*}{dr_2}$ is obtained using the implicit function theorem.

When $\mu = 0$, $1 - \frac{1}{2}\int_0^{\theta^*}\frac{L}{1-k}d\theta - \frac{1}{2}\int_{\theta^*}^2 qr_2 d\theta > 0$, i.e., (7) is not binding and r_2 solves (11) in the proposition. Since $1-k > L$ and $q^* \le 1$, r_2 must be greater than 1 for (7) to be satisfied. When $\mu > 0$,

$$1 - \frac{1}{2}\int_0^{\theta^*}\frac{L}{1-k}d\theta - \frac{1}{2}\int_{\theta^*}^2 qr_2 d\theta = 0$$

gives the optimal r_2, which again is greater than 1 in order for (7) to hold. The Lagrange multiplier μ is then pinned down by (32) and is equal to

$$\mu = \frac{\frac{\partial\theta^*}{\partial r_2}\left[R\theta^* - (1-k)\, r_2\right] + \int_{\theta^*}^{2} (1-k)\, d\theta}{\int_{\theta^*}^{2} q d\theta - \left[\frac{\partial\theta^*}{\partial r_2} + \frac{\partial\theta^*}{\partial q}\frac{dq}{dr_2}\right]\left[qr_2 - \frac{L}{1-k}\right]}.$$

Hence, the proposition follows. □

Proof of Proposition 3: To characterize the run thresholds $\underline{\theta}_x$ and θ_x^*, we follow the same steps as in the proof of Proposition 1. We start characterizing the range of fundamentals in which running is a dominant strategy. The threshold $\underline{\theta}_x$ is the solution to

$$R(\theta + x_1) - (1-k)\, r_2 = 0,$$

and is equal to

$$\underline{\theta}_x = \frac{(1-k)\, r_2}{R} - x_1 = \underline{\theta} - x_1.$$

For any $\theta \geq \underline{\theta}_x$, a depositor expects to receive qr_2 at date 2 and 1 at date 1 if no depositors run. Since $qr_2 \geq 1$, running is never optimal when $\theta \geq \underline{\theta}_x$. When $\theta < \underline{\theta}_x$, a depositor expects to receive 0 at date 2. Thus, it is always optimal to run in this range. Relying on the result of Proposition 1, we have that $\underline{\theta}_x$ is the run threshold when $1-k \leq L$. It is easy to see that $\frac{\partial\underline{\theta}_x}{\partial x_1} = -1 < 0$ and $\frac{\partial\underline{\theta}_x}{\partial k} = -\frac{r_2}{R} < 0$.

We now characterize the run threshold θ_x^* in the case $1-k > L$. As in the proof of Proposition 1, we have that $\theta_x^* > \underline{\theta}_x$. The characterization of θ_x^* follows the same steps as in the proof of Proposition 1. The only difference in a depositor's indifference condition is the cutoff $\widehat{n}_x(\theta) > \widehat{n}(\theta)$, which is given by the solution to

$$R(\theta + x_1)\left(1 - n\frac{(1-k)}{L}\right) - (1-n)(1-k)\, r_2 = 0.$$

Hence, a depositor's indifference condition is equal to:

$$\int_0^{\widehat{n}_x(\theta)} qr_2 d\theta = \pi_1, \tag{33}$$

where π_1 is given in (31). After a few manipulations, condition (12) in the proposition is obtained. It follows immediately that $\theta_x^* < \theta^*$ for any $x_1 > 0$, and $\frac{\partial\theta^*}{\partial x_1} = -1 < 0$. Concerning the comparative statics with respect to k, L and q, since $\theta_x^* = \theta^* - x_1$, we have that $\frac{\partial\theta_x^*}{\partial k} < 0$, $\frac{\partial\theta_x^*}{\partial L} < 0$ and $\frac{\partial\theta_x^*}{\partial q} < 0$ as in the economy without guarantees. Hence, the proposition follows. □

Proof of Proposition 4: The guarantee x_2 has no effect on $q_x^R = \{\underline{q}_x, q_x^*\}$ as both the bank's profits and run thresholds are independent of x_2. Consider now the effect of x_1. We consider separately the case when $1-k \leq L$ and when $1-k > L$. We start from the former.

Differentiating (14) with respect to x_1 we obtain

$$
\begin{aligned}
&-\frac{1}{2}\frac{\partial \underline{\theta}_x}{\partial x_1}\left[R\left(\underline{\theta}_x + x_1\right) - (1-k)\,r_2 - R\left(\underline{\theta}_x + x_1\right)\left(1 - \frac{(1-k)}{L}\right)\right] \qquad (34)\\
&+\frac{1}{2}\int_0^{\underline{\theta}_x} R\left(1 - \frac{(1-k)}{L}\right)d\theta + \frac{1}{2}\int_{\underline{\theta}_x}^{1-x_1} R d\theta\\
&= \frac{1}{2}\left[R\left(\underline{\theta}_x + x_1\right)\frac{(1-k)}{L} - (1-k)\,r_2\right] + \frac{1}{2}\int_0^{\underline{\theta}_x} R\left(1 - \frac{(1-k)}{L}\right)d\theta + \frac{1}{2}\int_{\underline{\theta}_x}^{1-x_1} R d\theta,
\end{aligned}
$$

since $\frac{\partial \underline{\theta}_x}{\partial x_1} = -1$. For banks with k such that $1-k = L$, $\frac{d\underline{q}_x}{dx_1} > 0$ since the expression above simplifies to $\frac{1}{2}\int_{\underline{\theta}_x}^{1-x_1} R d\theta > 0$. The same applies to banks with $k = 1$ since (34) simplifies to

$$
+\frac{1}{2}\int_0^{\underline{\theta}_x|_{k=1}} R d\theta + \frac{1}{2}\int_{\underline{\theta}_x|_{k=1}}^{1-x_1} R d\theta > 0.
$$

For values of $k \in (1-L, 1)$, the expression in (34) can be rearranged as

$$
+\frac{1}{2}\left\{R\left[x_1\frac{(1-k)}{L} + 1 - x_1\right] - (1-k)\,r_2\right\}. \qquad (35)
$$

The expression above is linear in k. Hence, since (34) is linear, positive at $k = 1$ and $k = 1-L$, it follows that it must also be positive for any $k \in (1-L, 1)$.

Consider now the case in which $1-k > L$. Differentiating (15) with respect to x_1, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\frac{\partial \theta_x^*}{\partial x_1}q\left[R\left(\theta_x^* + x_1\right) - (1-k)\,r_2\right] + \frac{1}{2}\int_{\theta_x^*}^{1-x_1} R d\theta - \frac{1}{2}\frac{\partial^2 \theta_x^*}{\partial q \partial x_1}q\left[R\left(\theta_x^* + x_1\right) - (1-k)\,r_2\right]\\
&-\frac{1}{2}\frac{\partial \theta_x^*}{\partial q}q\frac{\partial \theta_x^*}{\partial x_1}R - \frac{1}{2}\frac{\partial \theta_x^*}{\partial q}qR.
\end{aligned}
$$

Since $\frac{\partial \theta_x^*}{\partial x_1} = -1$ and $\frac{\partial^2 \theta_x^*}{\partial q \partial x_1} = \frac{\partial^2 \theta_x^*}{\partial x_1 \partial q} - 0$, the expression above simplifies to

$$
+\frac{1}{2}q\left[R\left(\theta_x^* + x_1\right) - (1-k)\,r_2\right] + \frac{1}{2}\int_{\theta_x^*}^{1-x_1} R d\theta > 0,
$$

and the proposition follows. □

Proof of Corollary 1: From Proposition 3, we know that $\theta_x^R = \underline{\theta}_x$ when $1-k \leq L$ and $\theta_x^R = \theta_x^*$ when $1-k > L$. Given that $Pr\left(\theta > \theta_x^R\right) = \frac{\theta_x^R}{2}$, $\frac{\partial \gamma_x^R}{\partial x_1} > 0$ follows directly from $\frac{dq_x^R}{dx_1} > 0$, $\frac{\partial \theta_x^R}{\partial x_1} - 1 < 0$ and $\frac{\partial \theta_x^R}{\partial q} \leq 0$. □

Proof of Proposition 5: When $1-k \leq L$, the relevant run threshold is $\underline{\theta}_x^P$, which corresponds to the solution to

$$R(\theta + x_1) - (1-k)\, r_2 = 0,$$

since when θ falls below $\underline{\theta}_x^P$ depositors expect to receive

$$q\frac{Rx_1}{1-k} + 1 - q\frac{Rx_2}{1-k} < 1,$$

and so prefer to run. Hence, $\underline{\theta}_x^P = \underline{\theta}_x$ holds.

When $1-k > L$, the relevant run threshold is θ_x^{*P}. Following the same steps as in in the proof of Proposition 3, the threshold θ_x^{*P} is pinned down by a depositor's indifference condition, which corresponds to expression (18) in the proposition.

To complete the proof, we need to compute the effect of q, k, x_1, and x_2 on θ_x^{*P}. We do this by using the implicit function theorem. Denote as $f(x_1, x_2, q, k, \theta) = 0$ the indifference condition in (18). Thus,

$$\frac{d\theta_x^{*P}}{dq} = -\frac{\frac{\partial f(.)}{\partial q}}{\frac{\partial f(.)}{\partial \theta}}, \quad \frac{d\theta_x^{*P}}{dk} = -\frac{\frac{\partial f(.)}{\partial k}}{\frac{\partial f(.)}{\partial \theta}}, \quad \frac{d\theta_x^{*P}}{dx_1} = -\frac{\frac{\partial f(.)}{\partial x_1}}{\frac{\partial f(.)}{\partial \theta}}, \frac{d\theta_x^{*P}}{dx_2} = -\frac{\frac{\partial f(.)}{\partial x_2}}{\frac{\partial f(.)}{\partial \theta}}$$

The denominator

$$\frac{\partial f(.)}{\partial \theta} = \frac{\partial \widehat{n}_x(\theta)}{\partial \theta} q \left[r_2 - \frac{Rx_1\left(1 - \widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)} \right] > 0,$$

since $\widehat{n}_x(\theta) = \frac{R(\theta+x_1)-(1-k)r_2}{R(\theta+x_1)\frac{(1-k)}{L}-(1-k)r_2}$ and so $\frac{\partial \widehat{n}_x(\theta)}{\partial \theta} = \frac{R - R\widehat{n}_x(\theta)\frac{(1-k)}{L}}{R(\theta+x_1)\frac{(1-k)}{L}-(1-k)r_2} = \frac{R\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{R(\theta+x_1)\frac{(1-k)}{L}-(1-k)r_2} > 0.$ Hence, the signs of the effect of q, k, x_1, and x_2 on θ_x^* are given by the opposite sign of the respective numerators. We have the following:

$$\frac{\partial f(.)}{\partial q} = \int_0^{\widehat{n}_x(\theta)} r_2 dn - \int_0^{\overline{n}} \frac{Rx_2\left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn + \int_{\widehat{n}_x(\theta)}^{\overline{n}} \frac{Rx_1\left(1 - n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn > 0,$$

$$\frac{\partial f(.)}{\partial k} = \frac{\partial \widehat{n}_x(\theta)}{\partial k} q \left[r_2 - \frac{Rx_1\left(1 - \widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)} \right] + \int_{\widehat{n}_x(\theta)}^{\overline{n}} q\frac{Rx_1}{(1-n)(1-k)^2} dn +$$
$$\int_0^{\overline{n}} (1-q)\frac{Rx_2}{(1-n)(1-k)^2} dn - \int_{\overline{n}}^1 \frac{L}{(1-k)^2 n} dn.$$

The expression for $\frac{\partial f(.)}{\partial k}$ can be rearranged as

$$\frac{\partial f\left(.\right)}{\partial k} = \frac{\partial \widehat{n}_x\left(\theta\right)}{\partial k} q r_2 - \int_{\overline{n}}^{1} \frac{L}{\left(1-k\right)^2 n} dn - \frac{\partial \widehat{n}_x\left(\theta\right)}{\partial k} q \frac{R x_1 \left(1 - \widehat{n}_x\left(\theta\right) \frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta\right)\right)\left(1-k\right)} \tag{36}$$
$$+ \int_{\widehat{n}_x(\theta)}^{\overline{n}} q \frac{R x_1}{\left(1-n\right)\left(1-k\right)^2} dn + \int_0^{\overline{n}} \left(1-q\right) \frac{R x_2}{\left(1-n\right)\left(1-k\right)^2} dn,$$

where $\frac{\partial \widehat{n}_x(\theta)}{\partial k} = \frac{R\left(\theta^{*P}+x_1\right)\frac{(1-k)}{L}-(1-k)r_2}{R\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)} > 0$. From the proof of Proposition 3, we know that θ_x^* is decreasing in k. This derivative can be computed using the implicit function theorem from (33) as follows:

$$\frac{\partial \theta_x^*}{\partial k} = -\frac{\frac{\partial \widehat{n}_x(\theta)}{\partial k} q r_2 - \int_{\overline{n}}^{1} \frac{L}{(1-k)^2 n} dn}{\frac{\partial \widehat{n}_x(\theta)}{\partial \theta} q r_2} < 0,$$

which implies that $\frac{\partial \widehat{n}_x(\theta)}{\partial k} q r_2 - \int_{\overline{n}}^{1} \frac{L}{(1-k)^2 n} dn > 0$ and so sum of the first two terms in (36) is positive. A sufficient condition for $\frac{\partial f(.)}{\partial k} > 0$ and so for $\frac{\partial \theta^{*P}}{\partial k} < 0$ is that

$$\frac{\partial \widehat{n}_x\left(\theta\right)}{\partial k} q \frac{R x_1 \left(1 - \widehat{n}_x\left(\theta\right) \frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta\right)\right)\left(1-k\right)} < \int_{\widehat{n}_x(\theta)}^{\overline{n}} q \frac{R x_1}{\left(1-n\right)\left(1-k\right)^2} dn,$$

that is

$$\frac{q R x_1}{L\left(1-k\right)} \left[\frac{\partial \widehat{n}_x\left(\theta\right)}{\partial k} \frac{\left(1 - \widehat{n}_x\left(\theta\right) \frac{(1-k)}{L}\right)}{\left(1-\widehat{n}_x\left(\theta\right)\right)\left(1-k\right)} - \frac{1}{1-k} \int_{\widehat{n}_x(\theta)}^{\overline{n}} \frac{1}{\left(1-n\right)} dn \right] < 0.$$

Substituting the expression for $\frac{\partial \widehat{n}_x(\theta)}{\partial k}$, we can express the sufficient condition simply as:

$$\frac{1}{1-k} \left[\frac{R\left(\theta^{*P}+x_1\right)\frac{(1-k)}{L} - \left(1-k\right) r_2}{R\ \left(1-\widehat{n}_x\left(\theta\right)\right)} - \int_{\widehat{n}_x(\theta)}^{\overline{n}} \frac{1}{\left(1-n\right)} dn \right] < 0.$$

The inequality above holds because the integral $\int_{\widehat{n}_x(\theta)}^{\overline{n}} \frac{1}{(1-n)} dn$ is increasing in n and is greater than $\frac{1}{1-\widehat{n}_x(\theta)}$ and $\frac{R\left(\theta^{*P}+x_1\right)\frac{(1-k)}{L}-(1-k)r_2}{R} < 1$.

Concerning the effect of x_1 and x_2, we have:

$$\frac{\partial f\left(.\right)}{\partial x_1} = \frac{\partial \widehat{n}_x\left(\theta^*\right)}{\partial x_1} q \left[r_2 - \frac{R x_1 \left(1 - n\frac{(1-k)}{L}\right)}{\left(1-n\right)\left(1-k\right)} \right] + \int_{\widehat{n}_x(\theta)}^{\overline{n}} q \frac{R\left(1 - n\frac{(1-k)}{L}\right)}{\left(1-n\right)\left(1-k\right)} dn > 0,$$

and

$$\frac{\partial f\left(.\right)}{\partial x_2} = \int_0^{\overline{n}} \left(1-q\right) \frac{R\left(1 - n\frac{(1-k)}{L}\right)}{\left(1-n\right)\left(1-k\right)} dn > 0.$$

Thus, the proposition follows. □

Proof of Proposition 6: The signs of $\frac{dq^{*P}}{dx}$ and $\frac{dq^{*P}}{dx}$ are given by the signs of the sum of $\frac{dq^{*P}}{dx_1}+\frac{dq^{*P}}{dx_2}$ and $\frac{dq^{*P}}{dx_1}+\frac{dq^{*P}}{dx_2}$, respectively.

Consider, first, the case when $1-k\leq L$. The first order condition with respect to q is given by (14), which implies that the sign of $\frac{dq^{*P}}{dx_1}$ is equal to the sign of the expression in (34). As shown in the proof of Proposition 4, this is always positive. Furthermore, since (14) does not depend on x_2, we have $\frac{dq^{*P}}{dx_2}=0$ when $1-k\leq L$.

Consider now the case when $1-k>L$. The first order condition with respect to q is given in (19). As q^{*P} is an interior solution, using the implicit function theorem the signs of $\frac{dq^{*P}}{dx_1}$ and $\frac{dq^{*P}}{dx_2}$ are equal to the sign of the derivative of (19) with respect to x_1 and x_2, respectively. Differentiating (19) with respect to x_1 and x_2 we obtain

$$\begin{aligned}&-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial x_1}\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right]-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x_1}qR\\&+\frac{1}{2}\int_{\theta_x^{*P}}^{1-x_1}Rd\theta-\frac{1}{2}\frac{\partial^2\theta_x^{*P}}{\partial q\partial x_1}q\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right]-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}qR,\end{aligned}\tag{37}$$

and

$$\begin{aligned}&-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial x_2}\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right]-\frac{1}{2}\frac{\partial^2\theta_x^{*P}}{\partial q\partial x_2}q\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right]\\&-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x_2}qR,\end{aligned}\tag{38}$$

respectively. To establish their sign, recall that

$$\frac{\partial\theta_x^{*P}}{\partial x_1}=-1-\frac{\int_{\widehat{n}_x(\theta)}^{\overline{n}}\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}\left[r_2-\frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}<0,\tag{39}$$

and

$$\frac{\partial\theta^{*P}}{\partial x_2}=-\frac{\int_0^{\overline{n}}(1-q)\,\frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}q\left[r_2-\frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}<0.\tag{40}$$

From (39), it can be seen immediately that $\frac{\partial^2\theta_x^{*P}}{\partial x_1\partial q}=0$. Differentiating (40) with respect to q we obtain

$$\frac{\partial^2\theta^{*P}}{\partial x_2 \partial q} = -\frac{-\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn + \frac{\partial\theta^{*P}}{\partial x_2}\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}q\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}$$

$$= \frac{\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}q\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} - \frac{1}{q}\frac{\partial\theta^{*P}}{\partial x_2}$$

$$= -\frac{\partial\theta^{*P}}{\partial x_2}\frac{1}{1-q} - \frac{\partial\theta^{*P}}{\partial x_2}\frac{1}{q} = -\frac{\partial\theta^{*P}}{\partial x_2}\frac{1}{(1-q)\,q} > 0.$$

Hence, we can rearrange (37) and (38) as follows:

$$-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial x_1}\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right] - \frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x_1}qR + \frac{1}{2}\int_{\theta_x^{*P}}^{1-x_1} Rd\theta - \frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}qR, \tag{41}$$

and

$$-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial x_2}\left[1-\frac{1}{1-q}\right]\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right] - \frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}\frac{\partial\theta_x^{*P}}{\partial x_2}qR < 0. \tag{42}$$

Since the expression in (42) negative, it follows that $\frac{dq^{*P}}{dx_2} < 0$.

To compute the overall effect of x on q, we sum up the terms in (37) and (38). After rearranging, the sign of $\frac{dq^{*P}}{dx}$ is given by the sign of

$$-\frac{1}{2}\left[\frac{\partial\theta_x^{*P}}{\partial x_1}+\frac{\partial\theta_x^{*P}}{\partial x_2}+\frac{\partial\theta_x^{2*P}}{\partial q\partial x_2}q\right]\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right]$$
$$+\frac{1}{2}\int_{\theta_x^{*P}}^{1-x_1} Rd\theta - \frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}qR\left[\frac{\partial\theta_x^{*P}}{\partial x_1}+\frac{\partial\theta_x^{*P}}{\partial x_2}+1\right].$$

Rearranging $\frac{\partial\theta_x^{*P}}{\partial x_1} = \frac{q}{1-q}\frac{\partial\theta_x^{*P}}{\partial x_2} - 1 + \frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}\left[r_2-\frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}$ and $\frac{\partial\theta_x^{2*P}}{\partial q\partial x_2} = \frac{\partial\theta_x^{*P}}{\partial x_2}\frac{1}{1-q} - \frac{\partial\theta^{*P}}{\partial x_2}\frac{1}{q}$, the expression can be written as

$$-\frac{1}{2}\left[\frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} - 1\right]\left[R\left(\theta_x^{*P}+x_1\right)-(1-k)\,r_2\right] + \frac{1}{2}\int_{\theta_x^{*P}}^{1-x_1} Rd\theta \tag{43}$$
$$-\frac{1}{2}\frac{\partial\theta_x^{*P}}{\partial q}qR\left[-\frac{\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}q\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} + \frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial\widehat{n}_x(\theta)}{\partial\theta}\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]}\right].$$

First, given that $\overline{n} > \widehat{n}_x(\theta)$ and $q < 1$, one can see that

$$-\frac{\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial \widehat{n}_x(\theta)}{\partial \theta} q \left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} + \frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial \widehat{n}_x(\theta)}{\partial \theta} \left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} < 0,$$

which implies that the last term in (43) is negative since $\frac{\partial \theta_x^{*P}}{\partial q} < 0$.

When $x_1 = x_2 = 0$, the fraction in the first bracket in (43) can be rewritten as

$$\frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial \widehat{n}_x(\theta)}{\partial \theta} \left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} = \frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\frac{R\theta_x^{*P}\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)} r_2}{R(\theta_x^{*P}+x_1)\frac{(1-k)}{L} - (1-k)r_2}} =$$

$$\frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\theta_x^{*P}\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{R\theta_x^{*P}\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}} \frac{\left[R\theta_x^{*P}\frac{(1-k)}{L} - (1-k)\, r_2\right]}{r_2} > 1,$$

since $\frac{\int_0^{\widehat{n}_x(\theta)} \frac{R\theta_x^{*P}\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{R\theta_x^{*P}\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}} > 1$ and $\frac{R\theta_x^{*P}\frac{(1-k)}{L}}{r_2} - (1-k) = \frac{(1-k)}{L}\left[\frac{R\theta_x^{*P}}{r_2} - L\right] > 1$ given that $L < 1-k$ and $R\theta_x^{*P} > (1-k)\, r_2$. Furthermore, since $\frac{\partial^2 \theta_x^{*P}}{\partial q \partial x_2} \equiv \frac{\partial^2 \theta_x^{*P}}{\partial x_2 \partial q} = \frac{2q-1}{(1-q)q}\frac{\partial \theta_x^{*P}}{\partial x_2}$, we can infer that $\frac{\partial \theta_x^{*P}}{\partial q} = \frac{2q-1}{(1-q)q}\theta_x^{*P}$. This implies that

$$\frac{1}{2}\frac{2q-1}{(1-q)\,q}\frac{\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn}{\frac{\partial \widehat{n}_x(\theta)}{\partial \theta}\left[r_2 - \frac{Rx_1\left(1-\widehat{n}_x(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_x(\theta))(1-k)}\right]} < \frac{2q}{1-q}\frac{\partial \theta_x^{*P}}{\partial x_2},$$

Hence, the first and last terms in (43) are negative, while the second one is positive.

Recall that θ_x^{*P} is strictly decreasing in k and we have assumed that there exists a cutoff value for k, denoted as $\widehat{k} > 0$, for which $\theta_x^{*P} = 1$ for $x = 0$. When $k = \widehat{k}$, the expression in (43) is negative and, by continuity, it continues to be negative around $\widehat{k}$. Similarly, given that when $1 - k = L$ the entire expression is positive, it follows that in the range $k \in \left[\widehat{k}, 1-L\right)$, there exists another cutoff $\widehat{k}_x^P$, such that $\frac{dq^{*P}}{dx} < 0$ for $k < \widehat{k}_x^P$. Hence, the proposition follows. □

Proof of Proposition 7: The proof proceeds in steps: First, we characterize depositors' withdrawal behavior. Then, we solve for the optimal q and characterize the effect of the introduction of the guarantees on the bank's monitoring choice. In doing so, we distinguish between the case in which the guaranteed amount is lost in bankruptcy and when it is protected from bankruptcy. We start with the former.

The characterization of depositors' withdrawal decision follows the same steps as in the proof of Propositions 3 and 5. Running is a dominant strategy when $\theta < \underline{\theta}_y$, which corresponds to the solution to

$$R\left[\theta + (1-\theta)\, y\right] - (1-k)\, r_2 = 0,$$

which gives

$$\underline{\theta}_y = \frac{\underline{\theta} - y}{1-y},$$

with $\underline{\theta} = \frac{(1-k)r_2}{R}$ corresponding to the run threshold when there are no guarantees.

When $1-k > L$, banks are exposed to panic runs. Following the same steps as in the previous sections, the condition pinning down θ_y^* is

$$\int_0^{\widehat{n}_y(\theta)} q r_2 dn = \pi_1,$$

where π_1 is given in (31) and $\widehat{n}_y(\theta)$ solves

$$R\left[\theta + (1-\theta)\, y\right]\left(1 - \frac{n\,(1-k)}{L}\right) - (1-n)\,(1-k)\, r_2 = 0.$$

$$\widehat{n}_y(\theta) = L \frac{r_2 - R\theta - k r_2 - Ry + Ry\theta}{(k-1)\,(R\theta - L r_2 + Ry - Ry\theta)}.$$

After a few manipulations, we obtain the expression in the proposition,

$$\theta_y^* = \frac{\theta^* - y}{1-y},$$

where θ^* corresponds to the run threshold when there are no guarantees, as given in (5). As shown in the proof of Proposition 1, $\underline{\theta}_y$ and θ_y^* are the relevant run thresholds for banks with high capital (i.e., $1-k \leq L$) and low capital ($1-k > L$), respectively.

We now move on to the choice of q. When $1-k \leq L$, the bank solves the following problem:

$$\max_q \frac{1}{2} q \int_0^{\underline{\theta}_y} R\left[\theta + (1-\theta)\, y\right]\left(1 - \frac{(1-k)}{L}\right) d\theta + \frac{1}{2} q \int_{\underline{\theta}_y}^1 \left[R\left[\theta + (1-\theta)\, y\right] - (1-k)\, r_2\right] d\theta$$
$$+ \frac{1}{2} q \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - \frac{cq^2}{2},$$

while when $1-k > L$, the objective function is

$$\max_q \frac{1}{2} q \int_{\theta_y^*}^1 \left[R\left[\theta + (1-\theta)\, y\right] - (1-k)\, r_2\right] d\theta + \frac{1}{2} q \int_1^2 \left[R - (1-k)\, r_2\right] d\theta - \frac{cq^2}{2}.$$

The first order condition for q is

$$\frac{1}{2}\int_0^{\underline{\theta}_y} R\left[\theta+(1-\theta)y\right]\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_y}^1\left[R\left[\theta+(1-\theta)y\right]-(1-k)r_2\right]d\theta$$
$$+\frac{1}{2}\int_1^2\left[R-(1-k)r_2\right]d\theta-cq=0,$$

when $1-k\leq L$, since $\frac{\partial\underline{\theta}_y}{\partial q}=0$ and

$$\begin{array}{c}\frac{1}{2}\int_{\theta_y^*}^1\left[R\left[\theta+(1-\theta)y\right]-(1-k)r_2\right]d\theta-\frac{\partial\theta_y^*}{\partial q}q\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right]\\ +\frac{1}{2}\int_1^2\left[R-(1-k)r_2\right]d\theta-cq=0,\end{array}\tag{44}$$

when $1-k>L$, with $\frac{\partial\theta_y^*}{\partial q}=\frac{1}{1-y}\frac{\partial\theta^*}{\partial q}<0$.

To compute the effect of y on the optimal q, we use the implicit function theorem. Thus, the sign of $\frac{dq_y}{dy}$ is equal to the sign of $\frac{\partial FOC_q}{\partial y}$. When $1-k\leq L$, $\frac{\partial FOC_q}{\partial y}$ is equal to:

$$\frac{1}{2}\int_0^{\underline{\theta}_y}R(1-\theta)\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_y}^1R(1-\theta)\,d\theta+\frac{1}{2}\frac{\partial\underline{\theta}_y}{\partial y}R\left[\underline{\theta}_y+\left(1-\underline{\theta}_y\right)y\right]\left(1-\frac{(1-k)}{L}\right).$$

The first two terms are positive, while the last one is negative since $\frac{\partial\underline{\theta}_y}{\partial y}=-\frac{1-\underline{\theta}_y}{1-y}$. When $1-k=L$, $\frac{\partial FOC_q}{\partial y}$ simplifies to $\frac{1}{2}\int_{\underline{\theta}_y}^1R(1-\theta)\,d\theta>0$. As $k\to1$, then $\underline{\theta}_y\to0$ for any $y>\underline{\theta}$, and so $\frac{\partial\underline{\theta}_y}{\partial y}=0$, while for $y<\underline{\theta}_y$ and $y\to0$, the term $\underline{\theta}_y+\left(1-\underline{\theta}_y\right)y\to0$. It follows that $\frac{\partial FOC_q}{\partial y}>0$ for all $k\in(1-L,1)$, so that $\frac{dq_y}{dy}>0$ holds.

Consider now the case when $1-k>L$: $\frac{\partial FOC_q}{\partial y}$ is given by

$$\frac{1}{2}\int_{\theta_y^*}^1R(1-\theta)\,d\theta-\frac{\partial\theta_y^*}{\partial y}\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right]-\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)$$
$$-\frac{\partial\theta_y^*}{\partial q}qR\left(1-\theta_y^*\right)-\frac{\partial^2\theta_y^*}{\partial q\partial y}q\left[R\left[\theta_y^*+\left(1-\theta_y^*\right)y\right]-(1-k)r_2\right],$$

where $\frac{\partial^2\theta_y^*}{\partial q\partial y}=\frac{\partial^2\theta_y^*}{\partial y\partial q}=\frac{1}{1-y}\frac{\partial\theta_y^*}{\partial q}<0$. All terms in the expression above are positive except $-\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)<0$. Recall that $\frac{\partial\theta_y^*}{\partial y}=-\frac{1-\theta_y^*}{1-y}$. Then we can write

$$-\frac{\partial\theta_y^*}{\partial q}q\frac{\partial\theta_y^*}{\partial y}R(1-y)-\frac{\partial\theta_y^*}{\partial q}qR\left(1-\theta_y^*\right)=-\frac{\partial\theta_y^*}{\partial q}qR\left[-\left(1-\theta_y^*\right)+\left(1-\theta_y^*\right)\right]=0,$$

and it follows that $\frac{dq_y}{dy}>0$ when $1-k>L$.

We now move on to the case when the guarantee amount is protected from bankruptcy. The threshold for fundamental runs is still given by $\underline{\theta}_y$ as specified above. The threshold for panic runs θ_y^{*P}, instead, now solves:

$$\int_0^{\widehat{n}_y(\theta)}qr_2dn+\int_{\widehat{n}_y(\theta)}^{\overline{n}}q\frac{R(1-\theta)y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn+\int_0^{\overline{n}}(1-q)\frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)}dn=\pi_1,\tag{45}$$

where π_1 and $\widehat{n}_y(\theta)$ are as above and $\overline{n}$ is still equal to $\frac{L}{1-k}$.

As we perform our analysis for the case in which $y \to 0$, the expression in (45) is increasing in θ and decreasing in n, so the usual derivations to characterize the panic run threshold θ_y^{*P} apply. Using the implicit function theorem, we can compute

$$\frac{\partial \theta_y^{*P}}{\partial q} = -\frac{\int_0^{\widehat{n}_y(\theta)} r_2 dn + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)y\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - \int_0^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q \left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} < 0,$$

and

$$\frac{\partial \theta_y^{*P}}{\partial y} = \frac{-\frac{\partial \widehat{n}_y(\theta)}{\partial y}\left[qr_2 - q\frac{R(1-\theta)y\left(1-\widehat{n}_y\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y)(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - (1-q)\int_0^{\overline{n}} \frac{R\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q \left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} < 0.$$

Starting from $\frac{\partial \theta_y^{*P}}{\partial y}$, we can compute $\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = \frac{\partial^2 \theta_y^{*P}}{\partial y \partial q}$ as follows:

$$\begin{aligned}\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = &-\frac{\frac{\partial \widehat{n}_y(\theta)}{\partial y}\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn - \int_0^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q \left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} \\ &- \frac{\partial \theta_y^{*P}}{\partial y}\frac{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta}\left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q \left[r_2 - \frac{R(1-\theta)y\left(1-\widehat{n}_y(\theta)\frac{(1-k)}{L}\right)}{(1-\widehat{n}_y(\theta))(1-k)}\right] - q\int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{Ry\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn} > 0.\end{aligned}$$

When $y = 0$, the expression above simplifies to

$$\begin{aligned}\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} = &-\frac{1}{\frac{\partial \widehat{n}_y(\theta)}{\partial \theta} q r_2}\left[\frac{\partial \widehat{n}_y(\theta)}{\partial y} r_2 + \int_{\widehat{n}_y(\theta)}^{\overline{n}} \frac{R(1-\theta)\left(1-n\frac{(1-k)}{L}\right)}{(1-n)(1-k)} dn\right] \\ &- \frac{\partial \theta_y^{*P}}{\partial y}\frac{1}{q},\end{aligned}$$

from which we can see that $\frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} > \frac{\partial \theta_y^{*P}}{\partial y}$ when $y = 0$.

The FOC_q is still given by (44), so that the expression for $\frac{\partial FOC_q}{\partial y}$ when $y = 0$ is given by

$$\begin{aligned}&\frac{1}{2}\int_{\theta_y^{*P}}^1 R(1-\theta)\, d\theta - \frac{\partial \theta_y^{*P}}{\partial y}\left[R\theta_y^{*P} - (1-k) r_2\right] \\ &- \frac{\partial \theta_y^{*P}}{\partial q} q \frac{\partial \theta_y^{*P}}{\partial y} R - \frac{\partial^2 \theta_y^{*P}}{\partial q \partial y} q \left[R\theta_y^{*P} - (1-k) r_2\right] - \frac{\partial \theta_y^{*P}}{\partial q} q R\left(1 - \theta_y^{*P}\right).\end{aligned}$$

Again, all terms are positive except $-\frac{\partial\theta_y^{*P}}{\partial q}q\frac{\partial\theta_y^{*P}}{\partial y}R<0$. When $1-k=L$, we know that $\frac{\partial FOC_q}{\partial y}>0$ and so $\frac{dq_y^{*P}}{dy}>0$. Recall that there exists a cutoff $\widehat{k}$ such that $\theta_y^{*P}\to 1$ when $k=\widehat{k}$. When $k=\widehat{k}$, the expression for $\frac{\partial FOC_q}{\partial y}$ evaluated at $y=0$ simplifies to

$$-\frac{\partial\theta_y^{*P}}{\partial y}\left[R-(1-k)\,r_2\right]-\frac{\partial^2\theta_y^{*P}}{\partial q\partial y}q\left[R-(1-k)\,r_2\right]-\frac{\partial\theta_y^{*P}}{\partial q}q\frac{\partial\theta_y^{*P}}{\partial y}R.$$

Since $\frac{\partial^2\theta_y^{*P}}{\partial q\partial y}>\frac{\partial\theta_y^{*P}}{\partial y}$ when $y=0$, the expression above is negative, which implies that $\frac{dq_y^{*P}}{dy}<0$. Hence, there exists a cutoff $\widehat{k}_y^P$, with $\widehat{k}<\widehat{k}_y^P<1-L$ such that $\frac{dq_y^{*P}}{dy}<0$ for $k<\widehat{k}_y^P$, and the proposition follows. $\square$

Proof of Proposition 8: Given the expressions for GD_x and GD_y in (21) and (22) and evaluating (22) at $y=\underline{y}=\frac{x}{1-\max(\underline{\theta}-x,0)}$, the expression that determines which guarantee scheme is more costly is

$$Rx-\frac{Rx^2}{2}-Rx\left(\underline{\theta}-x\right)\frac{1-k}{L}\gtrless\frac{Rx}{2\left(1-\max\left(\underline{\theta}-x,0\right)\right)}-\frac{Rx}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\left(\underline{\theta}-x\right)\frac{1-k}{L}$$
$$+\frac{Rx}{2\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\frac{1-k}{L}\left(\underline{\theta}-x\right)^2,$$

which can be simplified as

$$\frac{Rx}{2}\left[(2-x)-\frac{1}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)}-\left(\underline{\theta}-x\right)\frac{1-k}{L}\left(2-\frac{2-\left(\underline{\theta}-x\right)}{\left(1-\max\left(\underline{\theta}-x,0\right)\right)}\right)\right]\gtrless 0. \tag{46}$$

When (46) equals zero, the two guarantee schemes are equally costly. Note that this is the case for $x=0$ and $x=1$. When $x=1$, $\max\left(\underline{\theta}-x,0\right)=0$ and $\underline{\theta}_x=\underline{\theta}-x=0$, so that the expression above simplifies to $\frac{R}{2}\left[1-1\right]=0$.

We need now to check whether $GD_x\gtrless GD_y$ for any $x\in(0,1)$. Differentiate (46) with respect to x:

$$\frac{R}{2L\left(x\theta-1\right)^2}\left(L-x^2\underline{\theta}^3+2x^3\underline{\theta}^2+2x\underline{\theta}^2-3x^2\underline{\theta}-2Lx+2Lx^2\underline{\theta}^2-2Lx^3\underline{\theta}^2\right.$$
$$\left.+kx^2\underline{\theta}^3-2kx^3\underline{\theta}^2-4Lx\underline{\theta}+4Lx^2\underline{\theta}-2kx\underline{\theta}^2+3kx^2\underline{\theta}\right). \tag{47}$$

Evaluating (47) at $x=0$ gives $\frac{L}{L}=1>0$, so that, while the difference in the two loan guarantee schemes is zero at $x=0$, it becomes positive as soon as x becomes positive.

We now show that the difference in (46) is concave everywhere, which implies that $GD_x>GD_y$ for any $x\in(0,1)$. Start by differentiating (47) with respect to x again to obtain

$$\frac{2}{L\left(x\underline{\theta}-1\right)^3}\left(L-(1-k)\right)\left(1+\underline{\theta}+3x^2\underline{\theta}^2-x^3\underline{\theta}^3-3x\underline{\theta}\right)+(1-k). \tag{48}$$

Since $(x\underline{\theta}-1)^3<0$, to show that GD_x-GD_y is concave for any $x\in(0,1)$, we need to show that the expression in parentheses is positive.

For $x=0$, the expression is clearly positive, meaning that the difference GD_x-GD_y is concave around $x=0$. A sufficient condition for the expression to be positive for any $x\in(0,1)$ is

$$1+\underline{\theta}+3x^2\underline{\theta}^2-x^3\underline{\theta}^3-3x\underline{\theta}>0.$$

This is equivalent to showing that

$$\frac{1+\underline{\theta}}{x\underline{\theta}}>-\left(3x\underline{\theta}-x^2\underline{\theta}^2-3\right). \tag{49}$$

Rewrite the LHS in (49) as $\frac{1}{x\underline{\theta}}+\frac{1}{x}$. From this, we can see that for any x, the value of the LHS is minimal at $\underline{\theta}=1$, and equal to $\frac{2}{x}$.

Consider now the RHS in (49). Differentiating it with respect to $\underline{\theta}$ gives:

$$2x^2\underline{\theta}-3x$$

This derivative is positive if $2x^2\underline{\theta}-3x>0\Leftrightarrow 2x\underline{\theta}>3\Leftrightarrow x\underline{\theta}>\frac{3}{2}$, which can never happen since both x and $\underline{\theta}$ are less than 1. Hence, the RHS must be strictly decreasing in $\underline{\theta}$, and is maximized at $\underline{\theta}=0$. For $\underline{\theta}=0$, the RHS equals 3. The same thing is true for x: the RHS is decreasing in x, so the maximum value the RHS can take is 3, which occurs for either $x=0$ or $\underline{\theta}=0$.

Now consider the LHS. The lowest value it can take, as a function of x, is $\frac{2}{x}$. For this expression to become smaller than 3, i.e., the largest the RHS can be, we need $x>\frac{2}{3}$. Note now that x can only be greater than $\frac{2}{3}$ if $\underline{\theta}$ is also greater than $\frac{2}{3}$. Since the RHS is decreasing in x and $\underline{\theta}$, the most the RHS can be if $\frac{2}{3}\le x\le\underline{\theta}$ is

$$\left(\frac{2}{3}\right)^2\left(\frac{2}{3}\right)^2-3\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)+3=1.8642$$

which is less than the LHS.

Fix now $x=1$. The lowest value that the LHS can take when $x=1$ is 2. This is bigger than the value that the RHS takes when $x=\frac{2}{3}$. Thus, since both the LHS and the RHS are monotonically decreasing in x, it follows that the LHS is greater than the RHS for any $x>0$. This implies, in turn, that the difference GD_x-GD_y is concave for any $x\in(0,1)$ and so it is always positive as stated in the proposition.

To complete the proof we need to determine the effect of the two guarantees schemes on q. To do so, we compare FOC_q under GS_x and GS_y. The former is equal to

$$\begin{array}{c} [c]c+\frac{1}{2}\int_0^{\underline{\theta}_x} R\left(\theta+x\right)\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_x}^{1-x}\left[R\left(\theta+x\right)-\left(1-k\right)r_2\right]d\theta \\ +\frac{1}{2}\int_{1-x}^{2}\left[R-\left(1-k\right)r_2\right]d\theta-cq=0, \end{array} \tag{50}$$

while the latter is equal to

$$\begin{array}{c} +\frac{1}{2}\int_0^{\underline{\theta}_y} R\left(\theta+y-\theta y\right)\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_y}^{1}\left[R\left(\theta+y-\theta y\right)-\left(1-k\right)r_2\right]d\theta \\ +\frac{1}{2}\int_{1}^{2}\left[R-\left(1-k\right)r_2\right]d\theta-cq=0, \end{array} \tag{51}$$

since under GS_y the bank accrues a (per unit return) on the non-liquidated units equal to $R\theta+R\left(1-\theta\right)y=R\left(\theta+y-\theta y\right)$ and $\frac{\partial\underline{\theta}_y}{\partial q}=\frac{\partial\underline{\theta}_x}{\partial q}=0$.

We now compare (50) and (51) evaluated at $y=\underline{y}$ so that $\underline{\theta}_y=\underline{\theta}_x$. Given that q_x and q_y are interior solutions, for $q_x>q_y$, it must be that

$$\frac{1}{2}\int_0^{\underline{\theta}_x} R\left(\theta+x\right)\left(1-\frac{(1-k)}{L}\right)d\theta+\frac{1}{2}\int_{\underline{\theta}_x}^{1-x}\left[R\left(\theta+x\right)-\left(1-k\right)r_2\right]d\theta+\frac{1}{2}\int_{1-x}^{1}\left[R-\left(1-k\right)r_2\right]d\theta$$
$$-\frac{1}{2}\int_0^{\underline{\theta}_y} R\left(\theta+y-\theta y\right)\left(1-\frac{(1-k)}{L}\right)d\theta-\frac{1}{2}\int_{\underline{\theta}_y}^{1}\left[R\left(\theta+y-\theta y\right)-\left(1-k\right)r_2\right]d\theta\Bigg|_{y=\underline{y}}>0.$$

After a few manipulations, we can rearrange the expression on the LHS of the inequality above as follows:

$$\int_0^{\underline{\theta}_x} R\left(x-\frac{x}{1+x-\underline{\theta}}+\theta\frac{x}{1+x-\underline{\theta}}\right)\left(1-\frac{(1-k)}{L}\right)d\theta+\int_{\underline{\theta}_x}^{1-x} R\left(x-\frac{x}{1+x-\underline{\theta}}+\theta\frac{x}{1+x-\underline{\theta}}\right)d\theta$$
$$+\int_{1-x}^{1} R\left(1-\frac{x}{1+x-\underline{\theta}}\right)\left(1-\theta\right)d\theta$$
$$=\int_0^{\underline{\theta}_x}\frac{Rx^2}{1+x-\underline{\theta}}\left(1-\frac{(1-k)}{L}\right)d\theta+\int_{\underline{\theta}_x}^{1-x}\frac{Rx^2}{1+x-\underline{\theta}}d\theta+\int_{1-x}^{1}\frac{R}{1+x-\underline{\theta}}\left(1-\underline{\theta}\right)\left(1-\theta\right)d\theta>0.$$

Hence, since $FOC_{q_x}>FOC_{q_y}$, it follows that $q_x>q_y$, as desired. □

Proof of Proposition 9: Since $\delta<1$, the threshold for fundamental runs is the same as in the case without guarantees. This is due to the fact that when the bank is insolvent depositors receive $\delta<1$, but this is not enough to convince them not to run. Hence, for highly capitalized banks, when $1-k\leq L$, $\underline{\theta}_\delta$ is still given by (4).

Applying the same arguments as in the proof of Proposition 1, for banks with $1-k>L$, the relevant crisis threshold θ^*_δ corresponds to the solution to

$$\int_0^{\widehat{n}(\theta)} qr_2dn+\int_{\widehat{n}(\theta)}^1 q\delta dn+\int_0^1\left(1-q\right)\delta dn=\pi_1,$$

or, equivalently,

$$q \int_0^{\widehat{n}(\theta)} (r_2 - \delta)\, dn + \int_0^1 \delta dn = \pi_1,$$

where both $\widehat{n}(\theta)$ and π_1 are the same as in the case without guarantees. Following the same steps as in the proof of Proposition 1, we obtain the expression (23) in the proposition.

To complete the proof, we need to compute

$$\begin{aligned} \frac{\partial \theta_\delta^*}{\partial q} &= \frac{\frac{(1-k)r_2}{R}(r_2-\delta)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)} \left[1 - \frac{(qr_2 - \pi_1) + \delta(1-q)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)}\right] \\ &= -\frac{(r_2-\delta)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)} \left(\theta_\delta^* - \underline{\theta}\right), \end{aligned} \tag{52}$$

and

$$\begin{aligned} \frac{\partial \theta_\delta^*}{\partial \delta} &= \frac{\left[\frac{(1-k)r_2}{R}(1-q) - \frac{(qr_2-\pi_1)+\delta(1-q)}{\left(qr_2-\pi_1\frac{1-k}{L}\right)+\delta\left(\frac{1-k}{L}-q\right)}\left(\frac{1-k}{L} - q\right)\right]}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)} \\ &= -\frac{1}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)} \left(\theta_\delta^* \left(\frac{1-k}{L} - q\right) - \underline{\theta}(1-q)\right) < 0. \end{aligned} \tag{53}$$

Hence, the proposition follows. □

Proof of Proposition 10: When $1-k \le L$, the run threshold $\underline{\theta}_\delta$ is not affected by the deposit insurance δ as shown in the proof of Proposition 9. Hence, $\underline{q}_\delta$ is not affected by δ.

Consider now the case where $1-k > L$. In this case, the run threshold is θ_δ^* as characterized in (23). We use the implicit function theorem to compute $\frac{dq_\delta^*}{d\delta}$. Denote the expression in (24) as $FOC_{q_\delta^*} = 0$. It follows that:

$$\frac{dq_\delta^*}{d\delta} = -\frac{\frac{\partial FOC_{q_\delta^*}}{\partial \delta}}{\frac{\partial FOC_{q_\delta^*}}{\partial q}}.$$

The denominator $\frac{\partial FOC_{q_\delta^*}}{\partial q} < 0$ as q_δ^* is an interior solution. Hence, the sign of $\frac{dq_\delta^*}{d\delta}$ is equal to the sign of

$$\frac{\partial FOC_{q_\delta^*}}{\partial \delta} = -\frac{1}{2}\frac{\partial \theta_\delta^*}{\partial \delta}\left[R\theta_\delta^* - (1-k)\, r_2\right] - \frac{1}{2}\frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta} q \left[R\theta_\delta^* - (1-k)\, r_2\right] - \frac{1}{2}\frac{\partial \theta_\delta^*}{\partial q} q \frac{\partial \theta_\delta^*}{\partial \delta} R$$

All terms in the expression for $\frac{\partial FOC_{q_\delta^*}}{\partial \delta}$ are negative except the first one. We show next that the first term is dominated by the second, so that overall $\frac{\partial FOC_{q_\delta^*}}{\partial \delta} < 0$. To do so, we need to show that $q\left|\frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta}\right| > \left|\frac{\partial \theta_\delta^*}{\partial \delta}\right|$. Recall that $\frac{\partial \theta_\delta^*}{\partial q}$ is given in (52). Differentiating $\frac{\partial \theta_\delta^*}{\partial q}$ with respect to δ we get

$$\frac{\partial^2 \theta_\delta^*}{\partial q \partial \delta} = A \frac{\partial \theta_\delta^*}{\partial \delta} - \frac{-1 - A\left(\frac{1-k}{L} - q\right)}{\left(qr_2 - \pi_1 \frac{1-k}{L}\right) + \delta\left(\frac{1-k}{L} - q\right)} \left(\theta_\delta^* - \underline{\theta}\right) > 0,$$

with $A \equiv \frac{(r_2-\delta)}{\left(qr_2-\pi_1\frac{1-k}{L}\right)+\delta\left(\frac{1-k}{L}-q\right)}$. Since $qA = q\frac{(r_2-\delta)}{q(r_2-\delta)-\frac{1-k}{L}(\pi_1-\delta)} > 1$, we have that $q\frac{\partial^2\theta^*_\delta}{\partial q\partial\delta} > \left|\frac{\partial\theta^*_\delta}{\partial\delta}\right|$. Thus, $\frac{dq^*_\delta}{d\delta} < 0$ and the proposition follows. □

Proof of Lemma 1: Substituting the expressions for θ_L^B and θ_L^{SP} from (25) and (26), respectively, it is easy to see that for any $0 \le k < 1$ and $0 < q < 1$, $\theta_L^B < \theta_L^{SP}$ holds as

$$\frac{L-(1-q)(1-k)r_2}{qR} < \frac{L}{qR}.$$

The rest of the Lemma follows since θ_L^B increases with k, while θ_L^{SP} does not depend on k. □

Proof of Lemma 2: Comparing θ_B^L with $\underline{\theta}$, we have that

$$\theta_L^B < \underline{\theta} \iff r_2(1-k) > L.$$

Given that the LHS in the inequality above is decreasing in k and $\theta_L^B < \underline{\theta}$ when $k = 1-L$ and $\theta_L^B > \underline{\theta}$ when $k = 1$, there exists a cutoff value $\overline{k}_L \in (1-L, 1)$ solving $\theta_L^B = \underline{\theta}$. Hence, the lemma follows. □

Proof of Lemma 3: Denote as k_L^{SP} the cutoff value of capital for which $\underline{\theta} = \theta_L^{SP}$. This is equal to

$$k_L^{SP} = 1 - \frac{L}{qr_2} \ge 1-L,$$

for any $qr_2 \ge 1$. Given that $\frac{\partial\underline{\theta}}{\partial k} < 0$, while $\frac{\partial\theta_L^{SP}}{\partial k} = 0$, it follows that $\underline{\theta} > \theta_L^{SP}$ for $k < k_L^{SP}$ and $\underline{\theta} \le \theta_L^{SP}$ for $k \ge k_L^{SP}$. From Proposition 2, we know that $qr_2 = 1$ when $1-k \le L$. Hence, it follows that $k_L^{SP} = 1-L$ and, in turn, $\underline{\theta} \le \theta_L^{SP}$ when $k \le 1-L$, while $\theta^* > \underline{\theta} > \theta_L^{SP}$ when $k > 1-L$. Using the result from Lemma 2 that $\theta_L^B > \theta^R$ for $k > \overline{k}_L$ and $\theta_L^B \le \theta^R$ for $k \le \overline{k}_L$, we obtain the result in the lemma. □

Proof of Proposition 11: When $1-k \le L$, the bank is exposed to fundamental runs only. The introduction of the loan guarantee reduces $\underline{\theta}_x$ and θ_{Lx}^B, while it does not affect the planner's threshold θ_L^{SP}. Hence, $\theta_L^{SP} - \max\left\{\theta_{Lx}^B, \underline{\theta}_x\right\}$ strictly decreases with x.

When $1-k > L$, the bank is exposed to panic runs and the run threshold θ_x^* strictly decreases with x and k. Since $\underline{\theta}_x < \underline{\theta}$ and they are both decreasing in k, $\theta_L^{SP} = \underline{\theta}_x$ when $k = k_{L_x}^{SP} \equiv 1 - \frac{L}{qr_2} - \frac{x}{r_2} < 1-L$. Hence, since $\theta_x^* > \underline{\theta}_x$, there exists a cutoff value $0 < \widetilde{k}_L < k_{L_x}^{SP} < 1-L$ such that $\theta_L^{SP} \le \theta_x^*$ when $k \le \widetilde{k}_L$ and $\theta_L^{SP} > \theta_x^*$ when $k > \widetilde{k}_L$. The cutoff $\widetilde{k}_L$ solves $\theta_L^{SP} = \theta_x^*$ Hence, the proposition follows. □

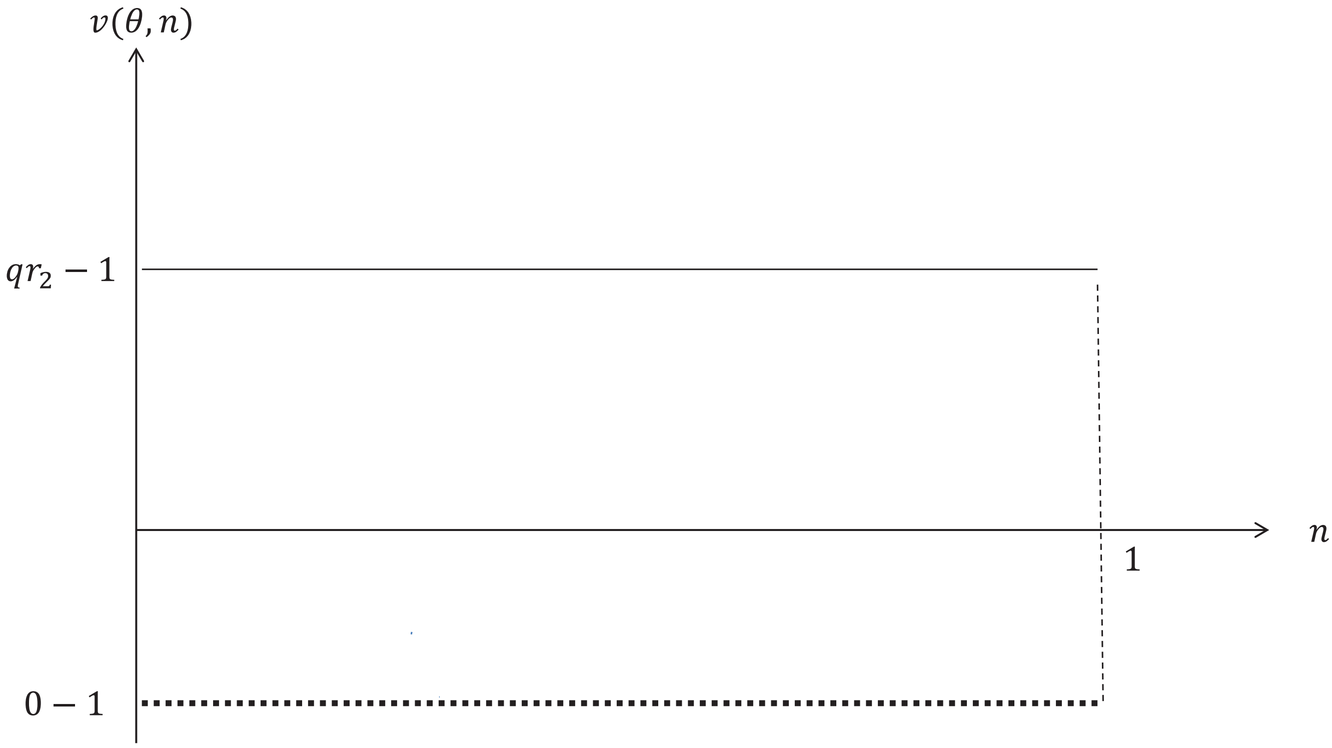


Figure 1: **Depositors' payoff differential when $(1-k) \leq L$.** The figure shows a depositor's payoff differential between withdrawing at date 2 and at date 1 as a function of the proportion of depositors withdrawing early, n, when 1-k ≤ L. The payoff differential is independent of n and only depends on whether the bank is solvent at date 2. The solid line represents the utility differential when the bank is solvent at date 2, while the dotted one captures the utility differential when the bank does not have enough resources to repay depositors even if no one runs.

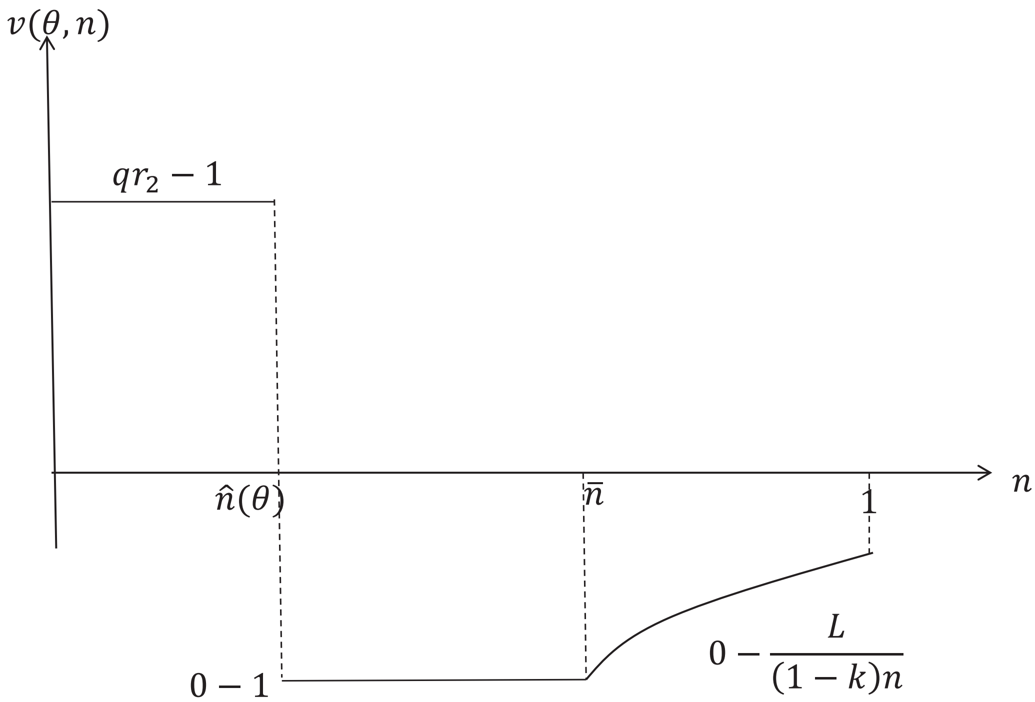


Figure 2: Depositors' payoff differential when $(1-k) > L$. The figure shows a depositor's payoff differential between withdrawing at date 2 and at date 1 as a function of the proportion of depositors withdrawing early, n, when 1-k > L. When $0 \leq n \leq \hat{n}(\theta)$, the bank has enough resources to make the promised repayments at both date 1 and 2. Thus, a depositor expects to receive r_2 with probability q at date 2, and 1 at date 1. When $\hat{n}(\theta) < n \leq \bar{\bar{n}}$, the bank can still pay 1 at date 1, while fails to repay depositors at date 2. Finally, when $\bar{\bar{n}} < n \leq 1$, the bank fails to make the promised repayment at both dates. Hence, a depositor receives nothing at date 2, while a share of the bank's liquidation proceeds at date 1.:

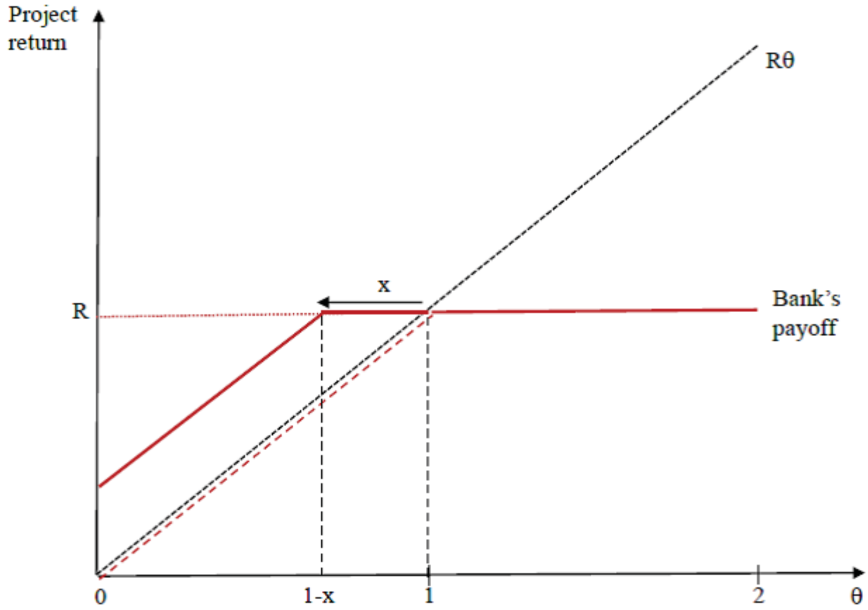


Figure 3: Bank payoff with a first-loss guarantee as a function of the fundamental $\boldsymbol{\theta}$. The figure shows the bank's payoff as a function of θ in the presence of a first-loss guarantee of size x. This is equal to $R(\theta + x)$ for $\theta < 1 - x$ and to R for $\theta \geq 1 - x$.

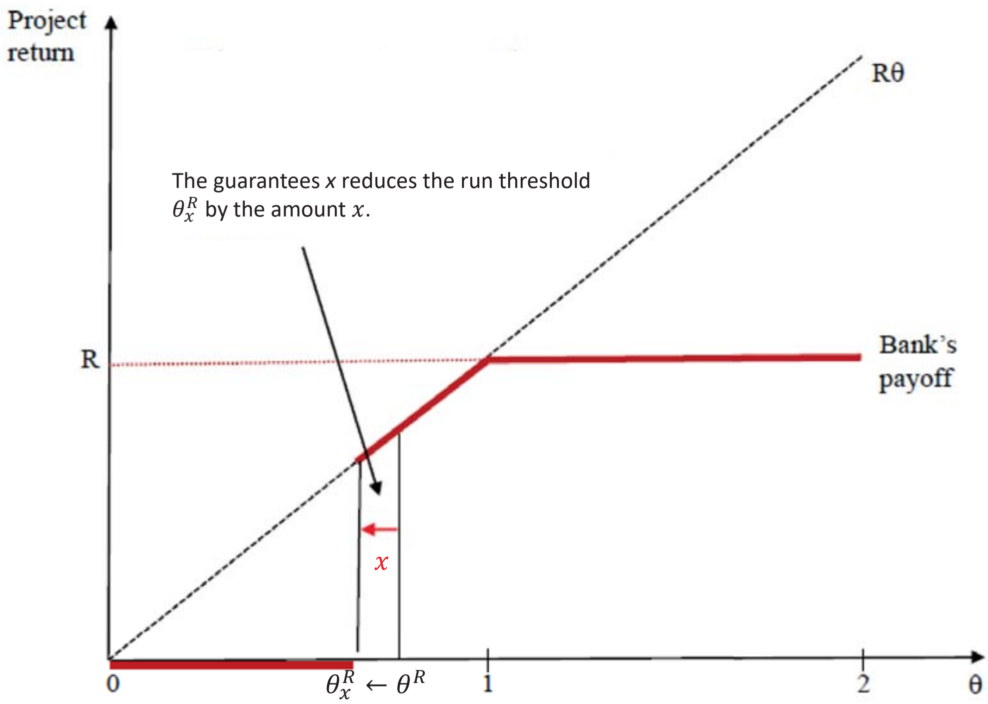
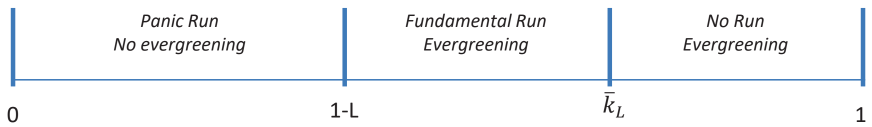


Figure 4: Run threshold with a first-loss guarantee of size x with full bankruptcy costs. The figure shows that the run threshold θ_x^R in the presence of a first-loss guarantee of size x decrease by an amount equal to x.

Economy without guarantees



Economy with a first-loss guarantee with full bankruptcy costs

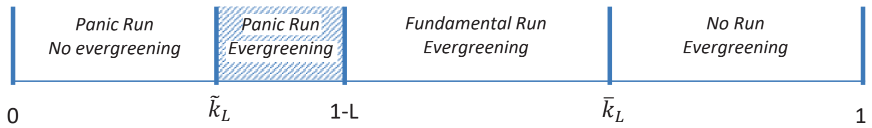


Figure 5: **Evergreening without and with a first-loss loan guarantee with full bankruptcy costs**. Without guarantees, banks with $k < 1 - L$ are subject to panic runs and do not engage in evergreening, while banks with $k > 1 - L$ do so both when they are subject to fundamental runs for $k\epsilon[1 - L, \tilde{k}_L]$, and when they are not subject to runs for $k\epsilon(\tilde{k}_L, 1]$. With a first-loss guarantee with full bankruptcy costs, also banks with capital $k\epsilon[\tilde{k}_L, 1 - L)$ start engaging in evergreening.